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On Diophantine equations involving difference of Lucas Numbers and powers of 2

Bilizimbéyé Edjeou[1] **and Amadou Tall**[2]

[1]Ecole Supérieure d'Informatique et de Gestion
Lomé Togo
[2]Faculté des Sciences et Technique
Université Cheikh Anta Diop
Dakar Sénégal
Email: edjeou.bilizimbeye@ugb.edu.sn, amadou7.tall@ucad.edu.sn

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Abstract

In this note, we find all positive integer solutions of the Diophantine equation $L_k - L_l = 2^t$ and $L_n - L_m = 2^{a_1} + 2^{a_2} + 2^{a_3}$, where $(L_n)_{n\geq 0}$ is the Lucas sequence. The tools used to solve our main theorem are linear forms in logarithms, properties of continued fractions, and a version of the Baker-Davenport reduction method in diophantine approximation.

1 Introduction

There is a vast literature on solving Diophantine equations involving the sequence $\{L_n\}_{n\geq 0}$ of Lucas numbers, the sequence $\left\{L_n^{(k)}\right\}_{n\geq 0}$ of k-generalized Lucas numbers or other recurrence sequences. The Lucas sequence $(L_k)_{k\geq 0}$ is a linear recurring sequence given by $L_0 = 2, L_1 = 1$ and

$$L_{k+2} = L_{k+1} + L_k.$$

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It follows the same recursive definition as the Fibonacci sequence $(F_k)_{k\geq 0}$ given by $F_0 = 0, F_1 = 1$ and

$$F_{k+2} = F_{k+1} + F_k, \quad \text{for } k \geq 2,$$

whose numbers are found everywhere in nature. The Fibonacci numbers are famous for possessing wonderful and amazing properties.

In 2014, Bravo and Luca [7] studied the Diophantine equation

$$L_k + L_l = 2^t$$

in positive integers k, l and t. Specifically, they proved the following theorems.

Theorem 1.1. *The only solutions* (k, l, t) *of the Diophantine equation* $L_k + L_l = 2^t$ *in positive integers* k, l, t *and with* $k \geq l$ *are*

$$(0, 0, 2); (1, 1, 1); (3, 3, 3); (2, 1, 2); (4, 1, 3); (7, 2, 5).$$

In 2020 [4] and 2021 [5], our work focused on the Diophantine equations $L_k + L_l + L_t = 2^d$ in non-negative integers k, l, t, d; and $L_k - 3^l = m$, where m is a fixed integer and k, l are positive variable integers. We provided all the solutions to these equations.

Similar equations involving Fibonacci and Padovan sequences are solved in [1, 14, 16, 17].

The most general result is due to Chim, Pink and Ziegler [11] who considered the case, where U_n and V_m are the $n-th$ and $m-th$ numbers in linear recurrence sequences $\{U_n\}_{n\geq 0}$ and $\{V_m\}_{m\geq 0}$ respectively and found effective upper bounds for $|c|$ such that the Diophantine equation

$$U_n - V_m = c.$$

In this paper, we extend this strategy and study the two Diophantine equations. We prove the following result.

Theorem 1.2. *All solutions* (n, m, b_1, b_2, b_3) *of the Diophantine equation*

$$L_n - L_m = 2^{b_1} + 2^{b_2} + 2^{b_3} \tag{1.1}$$

in non negative integers n, m, b_1, b_2 *and* b_3*, are*
$(4, 1, 1, 1, 1)$, $(5, 1, 1, 2, 2)$, $(5, 2, 1, 1, 2)$, $(6, 0, 2, 2, 3)$, $(6, 3, 1, 2, 3)$, $(7, 1, 2, 3, 4)$,

$(7,2,1,3,4)$, $(7,4,1,2,4)$, $(7,5,1,3,3)$, $(8,2,2,3,5)$, $(8,4,2,2,5)$, $(8,4,3,4,4)$, $(8,5,1,1,5)$, $(8,5,2,4,4)$, $(8,7,1,3,3)$, $(9,0,1,3,6)$, $(9,3,2,2,6)$, $(9,3,3,5,5)$, $(10,5,4,5,6)$, $(10,8,2,3,6)$, $(11,2,2,6,7)$, $(11,4,5,5,7)$, $(11,4,6,6,6)$, $(11,8,3,4,7)$, $(11,10,2,3,6)$, $(12,0,5,5,8)$, $(12,0,6,7,7)$, $(12,6,4,5,8)$, $(13,1,2,2,9)$, $(13,1,3,8,8)$, $(13,2,1,2,9)$, $(13,4,1,8,8)$, $(13,11,1,6,8)$, $(14,5,6,8,9)$, $(14,11,2,7,9)$, $(14,13,1,6,8)$, $(15,9,3,8,10)$, $(15,12,1,4,10)$, $(16,7,1,7,11)$, $(16,10,2,5,11)$.

Theorem 1.3. *All solutions* (k,l,t) *of the Diophantine equation*

$$L_k - L_l = 2^t \tag{1.2}$$

in non negative integers k,l *and* t*, are*

$$(0,1,0);(2,0,0);(3,2,0);(2,1,1);(3,0,1);(4,2,2);(5,4,2);(5,2,3);(6,0,4).$$

Our method of proof is similiar to the method described in [7].

2 Preliminaries

Before proceeding further, we recall the Binet formula for the Lucas numbers $(L_k)_{k\geq 0}$

$$L_k = \alpha^k + \beta^k, \quad \text{for} \quad k \geq 0,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

are the roots of the characteristic equation $x^2 - x - 1 = 0$. In particular, the inequality

$$\alpha^{k-1} \leqslant L_k \leqslant \alpha^{k+1} \tag{2.1}$$

holds for all $k \geq 0$.

To prove Theorem 1.3, using a result on linear forms in two logarithms, we require some notations. Let δ be an algebraic number of degree d with minimal polynomial

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d}(X - \delta^{(i)})$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the $\delta^{(i)}$ denotes the conjugates of δ. Then

$$h(\delta) = \frac{1}{d}(\log a_0 + \sum_{i=1}^{d} \log(\max\{|\delta^{(i)}|, 1\}))$$

is called the logarithmic height of δ. In particular, if $\delta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then

$$h(\delta) = \log \max\{|p|, q\}.$$

The following properties of the logarithmic height, will be used in the next section. Let δ, ν be algebraic numbers and $r \in \mathbb{Z}$. Then

- $h(\delta \pm \nu) \leq h(\delta) + h(\nu) + \log 2$,
- $h(\delta\nu^{\pm 1}) \leq h(\delta) + h(\nu)$,
- $h(\delta^r) = |r|h(\delta)$.

Using the above notation, we restate Laurent, Mignotte, and Nesterenko's result [15, Cor. 1].

Theorem 2.1. *Let* δ_1, δ_2 *be two non-zero algebraic numbers, and let* $\log \delta_1$ *and* $\log \delta_2$ *be any determinations of their logarithms. Set*

$$D = [\mathbb{Q}(\delta_1, \delta_2) : \mathbb{Q}]/[\mathbb{R}(\delta_1, \delta_2) : \mathbb{R}]$$

and

$$\Gamma := b_2 \log \delta_2 - b_1 \log \delta_1,$$

where b_1 *and* b_2 *are positive integers. Further, let* $A_1, A_2 > 1$ *be real numbers such that*

$$\log A_i \geq \max\{h(\delta_i), \frac{|h(\delta_i)|}{D}, \frac{1}{D}\}, \quad i = 1, 2.$$

Then, assuming that δ_1 and δ_2 are multiplicatively independent, we have

$$\log|\Gamma| > -30.9 \cdot D^4(\max\{\log b', \frac{21}{D}, \frac{1}{2}\})^2 \log A_1 \log A_2,$$

where

$$b' = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1}.$$

We also need the following general lower bound for linear forms in logarithms due to Matveev [18].

Theorem 2.2. *Assume that $\delta_1, \ldots, \delta_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D. Let $b_1, \ldots, b_n$ be rational integers, and*

$$\Lambda := \delta_1^{b_1} \cdots \delta_t^{b_t} - 1$$

be not zero. Then

$$|\Lambda| > \exp(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t),$$

where

$$B \geqslant \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geqslant \max\{Dh(\delta_i), |\log \delta_i|, 0.16\}, \quad \textit{for all} \quad i = 1, \ldots, t.$$

Finally, we present a version of the reduction method based on the Baker-Davenport Lemma [2], from Dujella and Pethő [12]. This will be one of the key tools used to reduce the upper bounds on the variables of the equation (1.2).

Lemma 2.1. *Let N be a positive integer, let p/q be a convergent of the irrational number γ such that $q > 6N$, and let A, B, μ be real numbers with $A > 0$ and $B > 1$. Define*

$$\xi := \|\mu q\| - N\|\gamma q\|,$$

where $\|\cdot\|$ denotes the distance to the nearest integer. If $\xi > 0$, then there is no solution to the inequality

$$0 < u\gamma - v + \mu < AB^{-w},$$

in positive integers u, v, and w, with $u \leqslant N$ and $w \geqslant \frac{\log(Aq/\xi)}{\log B}$.

3 The Proof of Theorem 1.2

Let us now get a relation between n and a_1. Combining (1.1) with the right inequality of (2.1), one gets that:

$$2^{a_1} < 2^{a_1} + 2^{a_2} + 2^{a_3} = L_n - L_m < \alpha^{n+1} - \alpha^{m-1} < \alpha^{n+1}$$

which leads to

$$a_1 < (n+1)\frac{\log \alpha}{\log 2}.$$

When $n \leq 400$, we have $a_1 \leq 278$. Then a brute force search with Sagemath in the range $0 \leq m < n \leq 400$ and $a_3 \leq a_2 \leq a_1 \leq 278$ gives the solutions in (1.1).

Thus, for the rest of our work, we assume that $n > 400$.

3.1 bounding n

Step 1. Show that

$$min\{(a_1 - a_2 + 2)\log 2, (m - n + 1)\log \alpha\} < 239 \cdot (\log 2n)^2 \quad \text{or}$$

$$min\{(a_1 - a_2 + 2)\log 2, (m - n + 1)\log \alpha\} < 2.63 \cdot 10^4.$$

Equation (1.1) can be rewritten as

$$\alpha^n + \beta^n - L_m = 2^{a_1} + 2^{a_2} + 2^{a_3}.$$

In the first step we consider n and a_1 to be large and by collecting "large" terms to the left hand side of the equation, we obtain

$$\begin{aligned} |\alpha^n - 2^{a_1}| &= |2^{a_2} + 2^{a_3} + L_m - \beta^n| \\ &< 2^{a_2} + 2^{a_3} + \alpha^{m+1} + 1 \\ &< \max\{2^{a_2+2}, 4 \cdot \alpha^{m+1}\}. \end{aligned}$$

Dividing by 2^{a_1} we get

$$\begin{aligned} \left|\alpha^{-n}2^{a_1} - 1\right| &< \max\{2^{a_2-n+2}, 2^{2-n} \cdot \alpha^{m+1}\} \\ &< \max\{2^2 \cdot 2^{a_2-n}, 2^{-n+2} \cdot \alpha^{m+1}\}. \end{aligned}$$

Hence we obtain the inequality

$$\left|\alpha^{-n}2^{a_1}-1\right| < \max\{2^{a_2-a_1+2}, \alpha^{m-n+2}\}. \tag{3.1}$$

In Step 1 we consider the linear form

$$\wedge = a_1 \log 2 - n \log \alpha.$$

Further, we put

$$\Gamma = e^{\wedge} - 1 = \alpha^n 2^{-a_1} - 1.$$

In order to apply Theorem 2.1, we take $\delta_1 := \alpha$, $\delta_2 := 2$, $b_1 := n$ and $b_2 := a_1$. Since $n > a_1$ we have $B = n$.

Note further that $h(\delta_1) = (\log \alpha)/2$ and $h(\delta_2) = \log 2$. Thus, we can choose $\log A_1 := \log \alpha$ and $\log A_2 := \log 2$.

Finally, by recalling that $a_1 \leq n$, we get

$$b' = \frac{n}{2\log 2} + \frac{a_1}{2\log \alpha} < 2n.$$

Since α and 2 are multiplicatively independent, we have, by Theorem 2.1 that,

$$\log \Gamma \geq -30.9 \cdot 2^4 \cdot (\max\{\log(2n), 21/2, 1/2\})^2 \cdot \log \alpha \cdot \log 2.$$

Thus

$$\log \Gamma \geq -165 \cdot (\max\{\log(2n), 21/2, 1/2\})^2 \tag{3.2}$$

and together with inequality 3.1 we have

$$min\{(a_1 - a_2 + 2)\log 2, (m - n + 1)\log \alpha\} < 239 \cdot (\log 2n)^2 \quad \text{or}$$

$$min\{(a_1 - a_2 + 2)\log 2, (m - n + 1)\log \alpha\} < 2.63 \cdot 10^4.$$

Thus we have proved so far:

Lemma 3.1. *Assume that (n, m, a_1, a_2, a_3) is a solution to equation* (1.1) *with $n \geq m \geq 0$ and $a_1 \geq a_2 \geq a_3 \geq 0$. Then we have*

$$\min\{(a_1 - a_2 + 2)\log 2, (m - n + 1)\log \alpha\} < 239 \cdot (\log 2n)^2 \quad \textit{or}$$

$$min\{(a_1 - a_2 + 2)\log 2, (m - n + 1)\log \alpha\} < 2.63 \cdot 10^4.$$

Now we have to distinguish between

Case 1: $min\{(a_1-a_2+2)\log 2,(m-n+1)\log\alpha\}=(a_1-a_2+2)\log 2$, *and*

Case 2: $min\{(a_1-a_2+2)\log 2,(m-n+1)\log\alpha\}=(m-n+1)\log\alpha$.

We will deal with these two cases in the following steps.

Step 2: We consider Case 1 and show that under the assumption that $(a_1-a_2+2)\log 2<239\cdot(\log 2n)^2$ or $(a_1-a_2+2)\log 2<2.63\cdot 10^4$, we obtain

$$min\{(a_1-a_3)\log 2,(n-m)\log\alpha\}<1.61\cdot 10^{15}(1+\log n)(\log(2n))^2.$$

Since we consider Case 1 we assume that

$$min\{(a_1-a_2+2)\log 2,(m-n+1)\log\alpha\}=(a_1-a_2+2)\log 2<239\cdot(\log 2n)^2$$

or $min\{(a_1-a_2+2)\log 2,(m-n+1)\log\alpha\}=(a_1-a_2+2)\log 2<2.63\cdot 10^4$.

By collecting "large" terms, i.e. terms involving n, m, a_1 and a_2, on the left hand side, we rewrite equation (1.1) as

$$|\alpha^n-2^{a_1}-2^{a_2}|=|2^{a_3}-\beta^n-\alpha^m+\beta^m|<2^{a_3}+\alpha^m+1$$

and obtain that

$$\left|a^n-2^{a_2}(2^{a_1-a_2}+1)\right|<2.2\cdot max\{2^{a_3},\alpha^m\}. \tag{3.3}$$

Dividing by a^n, we get by using inequality 3.3

$$\begin{aligned}\left|\alpha^{-n}2^{a_2}(2^{a_1-a_2}+1)-1\right|&<2\cdot max\{\alpha^{-n}\cdot 2^{a_3},\alpha^{m-n}\}\\&\le 2\cdot max\{2^{a_3-n},\alpha^{m-n}\}\end{aligned}$$

and obtain the inequality

$$\left|\alpha^{-n}2^{a_2}(2^{a_1-a_2}+1)-1\right|<2\cdot max\{2^{a_3-a_1},\alpha^{m-n}\}. \tag{3.4}$$

We shall apply Theorem 2.2 to inequality 3.4. Therefore we consider the following linear form in logarithms:

$$\wedge_1=-n\log\alpha+a_2\log 2+\log(2^{a_1-a_2}+1).$$

Further, we put

$$\Phi_1 = e^{\wedge_1} - 1 = \alpha^{-n} 2^{a_2}(2^{a_1-a_2}+1) - 1$$

and aim to apply Theorem 2.2 by taking

$$\alpha_1 = \alpha, \qquad \alpha_2 = 2, \qquad \alpha_3 = 2^{a_1-a_2}+1$$

$$b_1 = -n, \qquad b_2 = a_2, \qquad b_3 = 1.$$

Note that since $n > a_1 > a_2$ we have $B = n$. Next, we estimate the height of α_3 by using the properties of heights and Lemma (3.1):

$$\begin{aligned} h(\alpha_3) &\leq (a_1 - a_2)h(2) + \log 2 \\ &\leq (a_1 - a_2)\log 2 + \log 2 \\ &< 166 \cdot (\log(2n))^2 \quad \text{or} \quad 1.83 \cdot 10^4 \end{aligned}$$

which gives $h(\alpha_3) < 166 \cdot (\log(2n))^2$ or $h(\alpha_3) < 1.83 \cdot 10^4$. As before we have $h(\alpha_1) = \frac{1}{2}$ and $h(\alpha_2) = \log 2$. Now, we are ready to apply Theorem 2.2 and get

$$\log|\Phi_1| > C(1.2) \cdot \log 2 \cdot 83 \cdot (\log(2n))^2 > -1.61 \cdot 10^{14}(1+\log n)(\log(2n))^2, \tag{3.5}$$

or

$$\log|\Phi_1| > C(1.2) \cdot \log 2 \cdot \frac{1}{2} \cdot 1.83 \cdot 10^4 > -1.14 \cdot 10^{15}(1+\log n), \tag{3.6}$$

with $C(1.2) = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1+\log 2)(1+\log(n))$.

Combining those inequalities with inequality (3.4), we obtain

$$min\{(a_1 - a_3)\log 2, (n-m)\log\alpha\} < 1.61 \cdot 10^{14}(1+\log n)(\log(2n))^2$$

or

$$min\{(a_1 - a_3)\log 2, (n-m)\log\alpha\} < 1.14 \cdot 10^{15}(1+\log n).$$

Then

$$min\{(a_1 - a_3)\log 2, (n-m)\log\alpha\} < 1.61 \cdot 10^{15}(1+\log n)(\log(2n))^2. \tag{3.7}$$

At this stage, we have to consider two further subcases.

Case 1A: $min\{(a_1 - a_3)\log 2, (n-m)\log\alpha\} = (a_1 - a_3)\log 2$ and

Case 1B $min\{(a_1 - a_3)\log 2, (n-m)\log\alpha\} = (n-m)\log\alpha$

Step 3: We consider Case 1A and show that under the assumption that

$$a_1 - a_3 < 7.22 \cdot 10^{15}(1+\log n)(\log(2n))^2,$$

we obtain that

$$n - m < 7.1 \cdot 10^{27}(1+\log n)(\log(2n))^2.$$

In this step we consider n, a_1, a_2 and a_3 to be large. By collecting "large" terms on the left hand side we rewrite equation 1.1 as

$$|\alpha^n - 2^{a_1} - 2^{a_2} - 2^{a_3}| = |-\beta^n - \alpha^m + \beta^m| < \alpha^m + 1$$

and obtain that

$$\left|\alpha^n - 2^{a_1}(1 + 2^{a_2 - a_1} + 2^{a_3 - a_2})\right| < 1.2\alpha^m.$$

Dividing by a^n yields the inequality

$$\left|\alpha^{-n}2^{a_1}(1 + 2^{a_2 - a_1} + 2^{a_3 - a_1}) - 1\right| < 1.2\alpha^{m-n}. \tag{3.8}$$

We want to apply Theorem 2.2 to inequality 3.8 and consider the linear form

$$\wedge_A = -n\log\alpha + a_1\log 2 + \log((1 + 2^{a_2 - a_1} + 2^{a_3 - a_1})).$$

Further, we put

$$\Phi_A = e^{\wedge_A} - 1 = \alpha^{-n_1}2^{a_1}(1 + 2^{a_2 - a_1} + 2^{a_3 - a_1}) - 1$$

and aim to apply Theorem 2.2 with

$$\alpha_1 = \alpha, \qquad \alpha_2 = 2, \qquad \alpha_3 = (1 + 2^{a_2 - a_1} + 2^{a_3 - a_1})$$

$$b_1 = -n, \qquad b_2 = a_2, \qquad b_3 = 1.$$

Similarly as before we get that $B = n$. Next, let us estimate the height of α_3. Using the properties of heights, Lemma 3.1 and inequality 3.7, we get:

$$\begin{aligned} h(\alpha_3) &\leq (a_1 - a_2)h(2) + (a_1 - a_3)h(2) + \log 2 \\ &\leq (a_1 - a_2)\log 2 + (a_1 - a_3)\log 2 + \log 2 \\ &< 165.67 \cdot \log(2n) + 5.1 \cdot 10^{15}(1 + \log n)(\log(2n))^2 \\ &< 10^{16}(1 + \log n)(\log(2n))^2, \end{aligned}$$

which gives $h(\alpha_3) < 10^{16}(1+\log n)(\log(2n))^2$. As before we have $h(\alpha_1) = \frac{1}{2}$, $h(\alpha_2) = \log 2$ and $\phi_A \neq 0$. An application of Theorem 2.2 yields

$$\begin{aligned} \log|\Phi_A| &> \Delta_A \left(\frac{1}{2}\right)(\log 2)10^{16}(1 + \log n)(\log(2n))^2 \\ &> -3.37 \cdot 10^{27}(1 + \log n)(\log(2n))^2 \end{aligned}$$

where $\Delta_A = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2)$.

Combining this inequality with inequality 3.8 we obtain

$$n - m < 7.1 \cdot 10^{27}(1 + \log n)(\log(2n))^2. \tag{3.9}$$

Step 4: We consider Case 1B and show that under the assumption that

$$n - m < 10.4 \cdot 10^{15}(1 + \log n)(\log(2n))^2,$$

we obtain that

$$a_1 - a_3 < 2.710^{39}(1 + \log n)^2(\log(2n))^2.$$

By collecting large terms to the left hand side, where we consider n, m, a_1 and a_2 to be large, we rewrite equation 1.1 as

$$|\alpha^n - \alpha^m - 2^{a_1} - 2^{a_2}| = |2^{a_3} - \beta^n + \beta^m| < 2^{a_3} + 1$$

and obtain that

$$\left|\alpha^m(\alpha^{n-m}-1)-2^{a_2}(2^{a_1-a_2}+1)\right|<1.45\cdot 2^{a_3}.$$

Dividing by $2^{a_2}(2^{a_1-a_2}+1)$ we obtain the inequality

$$\left|\alpha^m 2^{-a_2}\left(\frac{\alpha^{n-m}-1}{2^{a_1-a_2}+1}\right)-1\right|<1.45\cdot 2^{a_3-a_1}. \tag{3.10}$$

We want to apply Theorem 2.2 to inequality 3.10. Hence we consider the linear form

$$\wedge_B = m\log\alpha - a_2\log 2+\log\left(\frac{\alpha^{n-m}-1}{2^{a_1-a_2}+1}\right).$$

Further, we put

$$\Phi_B = e^{\wedge_B}-1=\alpha^m 2^{-a_2}\left(\frac{\alpha^{n-m}-1}{2^{a_1-a_2}+1}\right)-1$$

and aim to apply Theorem 2.2 by taking

$$\alpha_1=\alpha,\qquad \alpha_2=2,\qquad \alpha_3=\frac{\alpha^{n-m}-1}{2^{a_1-a_2}+1}$$

$$b_1=m,\qquad b_2=-a_2,\qquad b_3=1$$

and get $B=n$ as in the steps before. Let us estimate the height of α_3. Using the properties of heights, Lemma 3.1 and inequalities 3.7, we get:

$$\begin{aligned} h(\alpha_3) &\leq (n-m)h(\alpha)+\log 2+(a_1-a_2)h(2)+\log 2\\ &=\frac{1}{2}(n-m)\log(\alpha)+(a_1-a_2)\log 2+2\log 2\\ &< 3.5\cdot 10^{27}(1+\log n)(\log(2n))^2+239\cdot(\log 2n)^2\\ &< 4\cdot 10^{27}(1+\log n)(\log(2n))^2, \end{aligned}$$

which gives $h(\alpha_3)<4\cdot 10^{27}(1+\log n)(\log(2n))^2$. A similar deduction as before yields $h(\alpha_1)=\log\alpha$, $h(\alpha_2)=\log 2$ and $\Phi_B\neq 0$. Now, we apply

Theorem 2.2 and get

$$\log|\Phi_B| > \Delta_B \cdot \log\alpha \cdot \log 2 \cdot 4 \cdot 10^{27}(1+\log n)^2(\log(2n))^2$$
$$> -1.3 \cdot 10^{39}(1+\log n)^2(\log(2n))^2,$$

where $\Delta_B = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1+\log 2)$.

Combining this inequality with inequality 3.10, we obtain

$$a_1 - a_3 < 2.7 \cdot 10^{39}(1+\log n)^2(\log(2n))^2. \tag{3.11}$$

Step 5: We consider Case 2 and show that under the assumption that

$$(n-m+1)\log\alpha < 239 \cdot (\log 2n)^2 \quad \text{or} \quad (n-m+1)\log\alpha < 2.63 \cdot 10^4,$$

we obtain

$$(a_1 - a_2)\log 2 < 9.7 \cdot 10^{38}(1+\log n)(\log 2n)^2.$$

Since we consider Case 2 we assume that

$$min\{(a_1-a_2)\log 2, (n-m)\log\alpha\} = (n-m)\log\alpha < 3.5 \cdot 10^2(\log 2n)^2.$$

In this step we consider n, m and a_1 to be large and by collecting "large" terms to the left hand side, we rewrite equation 1.1 as

$$|\alpha^n + \alpha^{n_2} - 2^{a_1}| = |2^{a_2} + 2^{a_3} + \beta^n + \beta^n| < 2 \cdot 2^{a_2} + 1$$

and obtain that

$$\left|\alpha^m(\alpha^{n-m}+1) - 2^{a_1}\right| < 2 \cdot 2^{a_2}.$$

Dividing through 2^{a_1} we get the inequality

$$\left|\alpha^m 2^{-a_1}\left(\alpha^{n-m}+1\right) - 1\right| < 2,45 \cdot 2^{(a_2-a_1)}. \tag{3.12}$$

Similarly as above we shall apply Theorem 2.2 to inequality 3.12. Hence we consider the linear form

$$\wedge_3 = m\log\alpha - a_1\log 2 + \log\left(\alpha^{n-m}+1\right).$$

Further, we put

$$\Phi_3 = e^{\wedge_3} - 1 = \alpha^m 2^{-a_1}\left(\alpha^{n-m}+1\right) - 1$$

and

$$\alpha_1 = \alpha, \qquad \alpha_2 = 2, \qquad \alpha_3 = \alpha^{n-m}+1,$$
$$b_1 = n_2, \qquad b_2 = -a_1, \qquad b_3 = 1.$$

Once again this choice yields $B = n$. Next, let us estimate the height of α_3. Using the properties of heights and Lemma 3.1 we find

$$\begin{aligned} h(\alpha_3) &\leq (n-m)h(\alpha) + \log 2 \\ &< (n-m)\log(\alpha) + \log 2 \\ &< 7.1 \cdot 10^{27}(1+\log n)(\log 2n)^2, \end{aligned}$$

which gives $h(\alpha_3) < 7.1 \cdot 10^{27}(1+\log n)(\log 2n)^2$. A similar deduction as before gives $h(\alpha_1) = \log\alpha$, $h(\alpha_2) = \log 2$ and $\Phi_2 \neq 0$. Thus by applying Theorem 2.2, we get

$$\begin{aligned} \log|\Phi_3| &> \Delta_3(\log 2)\cdot 7.1 \cdot 10^{27}(1+\log n)(\log 2n)^2 \\ &> -9.7 \cdot 10^{38}(1+\log n)(\log 2n)^2. \end{aligned}$$

with $\Delta_3 = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1+\log 2)$. Combining this inequality together with inequality 3.12, we obtain

$$(a_1 - a_2)\log 2 < 9.7 \cdot 10^{38}(1+\log n)(\log 2n)^2. \tag{3.13}$$

Step 6: We continue to consider Case 2 and show that under the assumption that

$$(n-m)\log\alpha < 2.61 \cdot 10^{13}\log n$$

and

$$(a_1 - a_2)\log 2 < 4.26 \cdot 10^{26}(\log n)^2,$$

we obtain

$$(a_1 - a_3)\log 2 < 6.73 \cdot 10^{50}(1+\log n)(\log 2n)^2.$$

We shall apply once more Theorem 2.2 to obtain an upper bound for $(a_1 -$

$a_3)\log 2$. The derivation is very similar to Case 1B. By collecting "large" terms on the left hand side, we rewrite equation 1.1 as

$$|\alpha^n + \alpha^m - 2^{a_1} - 2^{a_2}| = |2^{a_3} + \beta^n + \beta^m| < 2^{a_3} + 1.$$

By the same derivation as in Step 4 we obtain inequality (14), i.e.

$$\left|\alpha^{m_2 - a_2}\left(\frac{\alpha^{n-m}+1}{2^{a_1-a_2}+1}\right) - 1\right| < 1.3 \cdot 2^{a_3 - a_1}. \tag{3.14}$$

We have the same setting as in Case 1B, except that the estimate for the height of α_3 becomes

$$\begin{aligned} h(\alpha_3) &\le (n-m)h(\alpha) + \log 2 + (a_1 - a_2)h(2) + \log 2 \\ &= (n-m)\log(\alpha) + (a_1 - a_2)\log 2 + 2\log 2 \\ &< 7.1 \cdot 10^{27}(1+\log n)(\log(2n))^2 + 9.7 \cdot 10^{38}(1+\log n)(\log 2n)^2 \\ &< 10^{39}(1+\log n)(\log 2n)^2, \end{aligned}$$

which gives $h(\alpha_3) < 10^{39}(1+\log n)(\log 2n)^2$. Therefore by applying Theorem 2.2 similarly as before we obtain

$$(a_1 - a_3)\log 2 < 6.73 \cdot 10^{50}(1+\log n)(\log 2n)^2. \tag{3.15}$$

Lemma 3.2. *Assume that (n, m, a_1, a_2, a_3) is a solution to equation* (1.1) *with $n \ge m \ge 0$ and $a_1 \ge a_2 \ge a_3 \ge 0$. Then we have*

$n - m < 7.1 \cdot 10^{27}(1+\log n)(\log(2n))^2$, $a_1 - a_2 < 9.7 \cdot 10^{38}(1+\log n)(\log 2n)^2$ *and* $a_1 - a_3 < 6.73 \cdot 10^{50}(1+\log n)(\log 2n)^2$.

Step 7: We assume the bounds given in Lemma 3.2 and show that

$$n < 9.43 \cdot 10^{62}(1+\log n)(\log 2n)^2,$$

hence $n < 4 \cdot 10^{69}$.

We have to apply Theorem (2.2) once more. This time we rewrite equation 1.1 as

$$\left|\alpha^n(1+\alpha^{m-n}) - 2^{a_1}(1 + 2^{a_2-a_1} + 2^{a_3-a_1})\right| = |\beta^n + \beta^m| < 1.$$

Dividing by $\alpha^n(1+\alpha^{m-n})$ we obtain the inequality

$$\left|\alpha^{-n}2^{a_1}\left(\frac{1+2^{a_2-a_1}+2^{a_3-a_1}}{1+\alpha^{m-n}}\right)-1\right|<\alpha^{-n}. \tag{3.16}$$

In this final step we consider the linear form

$$\wedge_4=-n\log\alpha+a_1\log 2+log\left(\frac{1+2^{a_2-a_1}+2^{a_3-a_1}}{1+\alpha^{m-n}}\right).$$

Further, we put

$$\Phi_4=e^{\wedge_4}-1=\alpha^{-n}2^{a_1}\left(\frac{1+2^{a_2-a_1}+2^{a_3-a_1}}{1+\alpha^{m-n}}\right)-1.$$

We take

$$\alpha_1=\alpha,\qquad \alpha_2=2,\qquad \alpha_3=\alpha^{-n}2^{a_1}\left(\frac{1+2^{a_2-a_1}+2^{a_3-a_1}}{1+\alpha^{m-n}}\right),$$

$$b_1=-n,\qquad b_2=-a_1,\qquad b_3=1.$$

Thus we have $B=n$. By the results in Lemma 3.2 and similar computations done before we obtain

$$\begin{aligned}h(\alpha_3)&\leq(a_1-a_2)h(2)+(a_1-a_3)h(2)+(n-m)h(\alpha)+2\log 2\\&\leq(a_1-a_2)\log 2+(a_1-a_3)\log 2+(n-m)\log(\alpha)+2\log 2\\&<6.74\cdot 10^{50}(1+\log n)(\log 2n)^2,\end{aligned}$$

which gives $h(\alpha_3)<6.74\cdot 10^{50}(1+\log n)(\log 2n)^2$. As before we have $h(\alpha_1)=\log\alpha$, $h(\alpha_2)=\log 2$, and $\Phi_4\neq 0$. Now an application of Theorem (2.2) yields

$$\log|\Phi_4|>\Delta_4(\log 2)\left(6.74\cdot 10^{50}(1+\log n)(\log 2n)^2\right),$$

with $\Delta_4=-1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2\cdot(1+\log 2)$. Combining this inequality with inequality 3.16 we get

$$n<9.43\cdot 10^{62}(1+\log n)(\log 2n)^2,$$

which yields

$$n < 4 \cdot 10^{69}.$$

Lemma 3.3. *If* $(n,\ m,\ a_1,\ a_2,\ a_3)$ *is a solution to equation* (1.1) *with* $n \geq m \geq 0$ *and* $a_1 \geq a_2 \geq a_3 \geq 0$*, then we have*

$$a_1 < n < 4 \cdot 10^{69}.$$

3.2 Reduction of the bound

In this section, we will reduce the upper bound on n. Firstly, we determine a suitable upper bound on $n - m$, $a_1 - a_2$, $a_1 - a_3$, and later we use Lemma 2.1 to conclude that n must be smaller than 400.

Proof of Theorem 1.

Turning back to inequality (3.1), we obtain

$$0 < a_1 \log 2 - n \log \alpha < \max\{2^{a_2 - a_1 + 2}, \alpha^{m-n+2}\}.$$

Dividing across by $\log \alpha$, we get

$$0 < a_1 \gamma - n < \max\{8.32 \cdot 2^{a_2 - a_1}, 5.45 \cdot \alpha^{m-n}\}, \tag{3.17}$$

where

$$\gamma := \frac{\log 2}{\log \alpha}.$$

Let $[a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots] = [1, 2, 3, 1, 2, 3, 2, 4 \ldots]$ be the continued fraction expansion of γ, and let denote p_n/q_n its nth convergent. Recall also that $a_1 < 4 \cdot 10^{69}$. A quick inspection using Sagemath reveals that

$q_{140} < 4 \cdot 10^{69} < q_{141}$.

Furthermore, $a_N := \max\{a_i : i = 0, 1, \ldots, 44\} = a_{60} = 134$. So, from the known properties of continued fractions, we obtain that

$$|a_1 \gamma - n| > \frac{1}{(a_N + 2) a_1}. \tag{3.18}$$

Comparing estimates (3.18) and (3.17), we get right away that

$$\alpha^{n-m} < 5.45 \cdot 136 \cdot a_1 < 7.42 \cdot 10^{71} \quad \text{or} \quad 2^{a_1-a_2} < 8.32 \cdot 136 \cdot a_1 < 11.32 \cdot 10^{71} \tag{3.19}$$

leading to $n-m \leqslant 344$ or $a_1 - a_2 \leqslant 247$.

Step 1: We show that $a_1 - a_3 \leq 247$ or $n-m \leq 356$.

Let us start by considering inequality 3.4. Then we have the inequality

$$0 < \left| a_2 \cdot \frac{\log 2}{\log \alpha} - n + \frac{\log(2^{a_1-a_2}+1)}{\log \alpha} \right| < 4.16 \cdot max\{2^{a_3-a_1}, \alpha^{m-n}\}$$

and we apply the algorithm described in Remark 2 with

$$\gamma = \frac{\log 2}{\log \alpha}, \quad \mu = \frac{\log(2^{a_1-a_2}+1)}{\log \alpha}, \quad (A,B) = (4.16, 2)\, or\, (4.16, \alpha).$$

Let us be a bit more precise. We note that γ is irrational since 2 and α are multiplicatively independent, hence Lemma 2.1 is applicable. With $q = q_{142} > 6M$. This yields $\epsilon > 0.00073$ and therefore either $a_1 - a_3 \leq \frac{\log(4.16q/0.00073)}{\log 2} < 248$ or $n - m \leq \frac{\log(4.16q/0.00073)}{\log \alpha} < 357$.

Thus, we have either $a_1 - a_3 \leq 247$ or $n-m \leq 356$.

From this result we distinguish between

Case 1: $a_1 - a_3 \leq 247$ and

Case 2: $n - m \leq 356$.

Step 2: We consider Case 1 and show that under the assumption that $a_1 - a_2 \leq 247$ or $a_1 - a_3 \leq 247$, we have that $n - m \leq 344$.

In this step we consider inequality 3.8. Recall that

$$\wedge_A = -n \log \alpha + a_1 \log 2 + \log((1 + 2^{a_2-a_1} + 2^{a_3-a_1})).$$

Then we get

$$0 < \left| a_1 \cdot \frac{\log 2}{\log \alpha} - n + \frac{\log((1+2^{a_2-a_1}+2^{a_3-a_1}))}{\log \alpha} \right| < 2.5\alpha^{m-n}.$$

We apply the algorithm explained in Remark 2 again with the same γ and M as in Step 1, but now we choose $(A, B) = (2.5, \alpha)$ and

$$\mu = \frac{\log((1 + 2^{a_2-a_1} + 2^{a_3-a_1}))}{\log \alpha}$$

for each possible value of $a_1-a_2 = 0, 1, \ldots, 247$ and $a_1-a_3 = 0, 1, \ldots, 247$. In particular

$$q = q_{142}$$

is the largest denominator that appeared in applying our algorithm. Overall, we obtain $n - m \leq 355$. Within Case 1 we have to distinguish between two further sub-cases:

Case 1: $a_1 - a_2 \leq 247$ and

Case 2: $n - m \leq 355$.

Step 3: We consider Case 1A and show that under the assumption that $a_1 - a_2 \leq 247$ and $a_1 - a_3 \leq 247$, we have that $n - m \leq 356$.
In this step we consider inequality 3.10 and assume that $n_1 - n_2 \geq 20$. Recall that

$$\wedge_B = m \log \alpha - a_2 \log 2 + \log\left(\frac{\alpha^{n-m} - 1}{2^{a_1-a_2} + 1}\right).$$

Then we get

$$0 < \left| m \cdot \frac{\log \alpha}{\log 2} - a_2 + \frac{\log\left(\frac{\alpha^{n-m}-1}{2^{a_1-a_2}+1}\right)}{\log 2} \right| < 3.02 \cdot 2^{a_3-a_1}.$$

We proceed as in Remark 2 with the same γ and M as in Step 1, but we use $(A, B) = (3.02, 2)$ nstead. Moreover we consider

$$\mu = \frac{\log\left(\frac{\alpha^{n-m}-1}{2^{a_1-a_2}+1}\right)}{\log 2}$$

for each possible value of $a_1 - a_2 = 0, 1, \ldots, 247$ and $n - m = l = 0, 1, \ldots, 356$. As in the previous step we apply the algorithm Lemme 2.1 and start with the 142^{nd} convergent of γ as before and continue with the algorithm until a positive ϵ. Thus we can compute a new upper bound for

$a_1 - a_3$ by the formula $a_1 - a_3 < \frac{\log(3.02q/\epsilon)}{\log 2}$ for the respective choices of q and ϵ. Overall we obtain that

$$a_1 - a_3 \leq 357.$$

Step 4: We consider Case 1B and show that under the assumption that $n - m \leq 355$, we have that $a_1 - a_2 \leq 233$.

Turning back to inequality 3.12

$$\wedge_C = m \log \alpha - a_1 \log 2 + \log\left(\alpha^{n-m} + 1\right).$$

Then we get

$$0 < \left| m \cdot \frac{\log \alpha}{\log 2} - a_1 + \frac{\log\left(\alpha^{n-m} + 1\right)}{\log 2} \right| < 22 \cdot 2^{(a_2 - a_1)}.$$

We apply our algorithm with the same γ and M as in the previous steps, but we use $(A, B) = (22, 2)$ and

$$\mu = \frac{\log\left(\alpha^{n-m} + 1\right)}{\log 2}$$

for each possible value of $a_1 - a_2 = k = 0, 1, \ldots, 224$ and $n_1 - n_2 = r = 0, 1, \ldots, 324$. We run our algorithm starting with $q = q_{144}$ and compute the upper bound for $a_1 - a_3$ by the formula $a_1 - a_2 < \frac{\log(22q/\epsilon)}{\log 2}$ for respective choices of q and ϵ, provided the algorithm terminates. For those pairs (k, r) for which the algorithm terminates we obtain

$$a_1 - a_2 \leq 258.$$

Step 5: We consider Case 2 and show that under the assumption that $n - m \leq 355$ we have that $a_1 - a_2 \leq 224$. In this step we consider inequality 3.14 and assume that $a_1 - a_2, a_1 - a_3 \leq 20$. Recall that

$$\wedge_2 = \wedge_B = m \log \alpha - a_2 \log 2 + \log\left(\frac{\alpha^{n-m} + 1}{2^{a_1 - a_2} + 1}\right).$$

Then we get

$$0 < \left| m \cdot \frac{\log \alpha}{\log 2} - a_2 + \frac{\log\left(\frac{\alpha^{n-m}+1}{2^{a_1-a_2}+1}\right)}{\log 2} \right| < 1.9 \cdot 2^{a_3-a_1}.$$

We apply our algorithm with the same γ and M, but we use $(A, B) = (1.9, 2)$ and

$$\mu = \frac{\log\left(\frac{\alpha^{n-m}+1}{2^{a_1-a_2}+1}\right)}{\log 2},$$

for each possible value of $n - m = r = 0, 1, \ldots, 315$. Similar as in Step 4 we obtain $a_1 - a_3 \leq 249$.

Step 6: Under the assumption that $n-m \leq 356$, $a_1-a_2 \leq 247$ and $a_1-a_3 \leq 249$, we show that $n \leq 400$.

For the last step we consider inequality (22). Recall that

$$\wedge_3 = -n \log \alpha + a_1 \log 2 + log\left(\frac{1 + 2^{a_2-a_1} + 2^{a_3-a_1}}{1 + \alpha^{m-n}}\right)$$

and inequality (22) yields that $|\wedge_3| < 2.02\alpha^{-n_1}$. Then we get

$$0 < \left| a_1 \cdot \frac{\log 2}{\log \alpha} - n_1 + \frac{log\left(\frac{1+2^{a_2-a_1}+2^{a_3-a_1}}{1+\alpha^{m-n}}\right)}{\log \alpha} \right| < 2.1\alpha^{-n}.$$

We proceed as described in Remark 2 with the same γ and M as in the previous steps, but we use $(A, B) = (2.1, \alpha)$ and

$$\mu = \frac{log\left(\frac{1+2^{a_2-a_1}+2^{a_3-a_1}}{1+\alpha^{m-n}}\right)}{\log \alpha},$$

for each possible value of $a_1 - a_2 = k = 0, 1, \ldots, 247$, $a_1 - a_3 = l = 0, 1, \ldots, 249$ (with respect to the obvious condition that $a_1 - a_2 \leq a_1 - a_3$) and $n - m = r = 0, 1, \ldots, 355$. Starting with $q = q_{142}$ we compute the upper bound for n by the formula $n < \frac{\log(2.1q/\epsilon)}{\log \alpha}$ for the respective choices of q such that $\epsilon > 0$. For all triples $(n - m, a_1 - a_2, a_1 - a_3)$ the algorithm terminates and yields

$$n \leq 359.$$

This is false because our assumption is that $n > 400$. Thus, Theorem 1.2 is proven.

4 The Proof of Theorem 1.3

If $k \leq 200$, then a brute force search with Sagemath in the range $0 \leq l < k \leq 200$ and $(k, l) = (0, 1)$ gives the solutions:

$(0, 1, 0)$, $(2, 0, 0)$; $(3, 2, 0)$; $(2, 1, 1)$; $(3, 0, 1)$; $(4, 2, 2)$; $(5, 4, 2)$; $(5, 2, 3)$; $(6, 0, 4)$.

Thus, for the rest of the paper we assume that $k > 200$ and $k > 0$.

Let us now get a relation between k and t. Combining (1.2) with the right inequality of (2.1), one gets that:

$$2^t \leq 2\alpha^k - \alpha^{l-1} < 2^{k+1} - \alpha^{l-1} = 2^{k+1}(1 - 2^{-(k+1)}\alpha^{l-1}) \leq 2^{k+1}$$

which leads to $t \leqslant k$.

This estimate is essential for our purpose. On the other hand, we rewrite equation 1.2 as

$$\alpha^k - 2^t = -\beta^k + L_l. \tag{4.1}$$

We now take absolute values in the above relation obtaining

$$|\alpha^k - 2^t| \leq |\beta|^k + L_l < \frac{1}{2} + 2\alpha^l. \tag{4.2}$$

Dividing both sides of the above expression by α^k and taking into account that $k > l$, we get

$$|1 - 2^t\alpha^{-k}| < \frac{1}{2}\alpha^{-k} + 2\alpha^{-k+l} < 3\alpha^{-k+l}.$$

Thus

$$|1 - 2^t\alpha^{-k}| < \frac{3}{\alpha^{k-l}}. \tag{4.3}$$

In order to apply Theorem 2.1, we take $\delta_1 := \alpha$, $\delta_2 := 2$, $b_1 := k$ and $b_2 := t$.

So, $\Gamma := b_2 \log \delta_2 - b_1 \log \delta_1$, and therefore estimation (4.3) can be rewritten as

$$|1-e^{\Gamma}| < \frac{3}{\alpha^{k-l}}. \tag{4.4}$$

The algebraic number field containing δ_1, δ_2 is $\mathbb{Q}(\sqrt{5})$, so we can take $D := 2$. By using equation (1.2) and the Binet formula for the Lucas sequence, we have

$$\alpha^k = L_k - \beta^k < L_k + 1 \leq L_k + L_l = 2^t. \tag{4.5}$$

Consequently, $1 < 2^t\alpha^{-k}$ and so $\Gamma > 0$. This, together with (4.4), gives

$$0 < \Gamma < \frac{3}{\alpha^{k-l}} \tag{4.6}$$

where we have also used the fact that $\log(1+x) \leqslant x$ for all $x \in \mathbb{R}^+$.
Hence,

$$\log\Gamma < \log 3 - (k-l)\log\alpha. \tag{4.7}$$

Note further that $h(\delta_1) = (\log\alpha)/2$ and $h(\delta_2) = \log 2$. Thus, we can choose $\log A_1 := \log\alpha$ and $\log A_2 := \log 2$.
Finally, by recalling that $t \leq k$, we get

$$b' = \frac{k}{2\log 2} + \frac{t}{2\log\alpha} < 2k.$$

Since α and 2 are multiplicatively independent, we have, by Theorem 2.2 that,

$$\log\Gamma \geq -30.9 \cdot 2^4 \cdot (\max\{\log(2k), 21/2, 1/2\})^2 \cdot \log\alpha \cdot \log 2.$$

Thus

$$\log\Gamma > -174 \cdot (\max\{\log(2k), 21/2, 1/2\})^2. \tag{4.8}$$

We now combine (4.7) and (4.8) to obtain

$$(k-l)\log\alpha < 180 \cdot (\max\{\log(2k), 21/2\})^2. \tag{4.9}$$

Let us now get a second linear form in logarithms. To this end, we now rewrite equation (1.2) as follows:

$$\alpha^k(1-\alpha^{(l-k)}) - 2^t = \beta^l(1-\beta^{(k-l)}). \tag{4.10}$$

Taking absolute values in the above relation and using the fact that $\beta = (1-\sqrt{5})/2$

we get

$$|\alpha^k(1-\alpha^{(l-k)}) - 2^t| = |\beta^l(1-\beta^{(k-l)})| < 2|\beta|^l < 2 \tag{4.11}$$

for all $k > 200$ and $l \geq 0$. Dividing both sides of the above inequality by the first term of the left-hand side, we obtain

$$|1 - 2^t\alpha^{-k}(1-\alpha^{(l-k)})^{-1}| < \frac{2}{\alpha^k(1-\alpha^{(l-k)})} < \frac{10}{\alpha^k}. \tag{4.12}$$

We are now ready to apply Matveev's result in Theorem 2.2. To do this, we take the parameters $n := 3$ and

$\delta_1 := 2$, $\delta_2 := \alpha$, $\delta_3 := (1-\alpha^{(l-k)})$.

We take $b_1 := t$, $b_2 := -k$ and $b_3 := -1$. As before, $\mathbb{K} := \mathbb{Q}(\sqrt{5})$ contains $\delta_1, \delta_2, \delta_3$ and has $D := [\mathbb{K} : \mathbb{Q}] = 2$. To see why the left-hand side of (4.12) is not zero, note that otherwise, we would get the relation

$$\alpha^k - \alpha^l = 2^t. \tag{4.13}$$

From 1.2, we get

$$\beta^k - \beta^l = 0. \tag{4.14}$$

Further, we obtain

$$\beta^k = \beta^l.$$

This is impossible because $k \neq l$. Thus,

$$1 - 2^t\alpha^{-k}(1-\alpha^{(l-k)})^{-1}$$

is not zero. In this application of Matveev's theorem we take $A_1 := 2\log 2$ and $A_2 := \log\alpha$. Since $t \leq k$; it follows that we can take $B := k$. Let us now estimate $h(\delta_3)$. We begin by observing that

$\delta_3 = (1-\alpha^{(l-k)})$ and $\delta_3^{-1} < 3$.

So that

$$|\log\delta_3| < 1. \tag{4.15}$$

Next, notice that

$$h(\delta_3) \leq (k-l)\log\alpha + \log 2. \tag{4.16}$$

Hence, we can take

$$A_3 := 2 + (k-l)\log\alpha > \max\{2h(\delta_3), |\log\delta_3|, 0.16\}.$$

Now Matveev's theorem implies that a lower bound on the left-hand side of (4.12) is

$$\log|\Lambda| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2(1+\log 2)(1+\log(k))\cdot 2\log 2\cdot 2\log\alpha\cdot(2+(k-l)\log\alpha).$$

So, inequality (4.12) yields

$$k < 2.8\cdot 10^{12}\log(k)\cdot(2+(k-l)\log\alpha) \tag{4.17}$$

where we used the inequality $1+\log(k) < 2\log(k)$, which holds because $k > 200$.

Using now (4.9) in the right-most term of the above inequality (4.17) and performing the respective calculations, we arrive at

$$k < 5.1\cdot 10^{14}\log(k)(\max\{\log(2k), 21/2\})^2. \tag{4.18}$$

If $\max\{\log(2k), 21/2\} = 21/2$, then it follows from (4.18) that

$$k < 5.7\cdot 10^{16}\log(k)$$

giving

$$k < 2.5\cdot 10^{18}.$$

If $\max\{\log(2k), 21/2\} = \log(2k)$, then we see from (4.18) that

$$k < 5.1\cdot 10^{14}\log(k)(\log(2k))^2,$$

and so

$$k < 5\cdot 10^{19}.$$

In any case, we have that

$$k < 5\cdot 10^{19}.$$

We summarize what we have proved so far in the following lemma.

Lemma 4.1. *If (k, l, t) is a solution in positive integers of equation (1.2) with $k > l$ and $k > 200$, then inequalities*

$$t \le k < 5.10^{19}$$

hold.

4.1 The final computations

In this section, we will reduce the upper bound on k. Firstly, we determine a suitable upper bound on $k-l$, and later we use Lemma 2.1 to conclude that k must be smaller than 200. Turning back to inequality (4.6), we obtain

$$0 < t\log 2 - k\log\alpha < \frac{3}{\alpha^{k-l}}.$$

Dividing across by $\log\alpha$, we get

$$0 < t\gamma - k < \frac{7}{\alpha^{k-l}}, \tag{4.19}$$

where

$$\gamma := \frac{\log 2}{\log\alpha}.$$

Let $[a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots] = [1, 2, 3, 1, 2, 3, 2, 4 \ldots]$ be the continued fraction expansion of γ, and let denote p_n/q_n its nth convergent. Recall also that $t < 5\cdot 10^{19}$ by Lemma 4.1. A quick inspection using Sagemath reveals that

$$12744458107726027589 = q_{43} < 5\cdot 10^{19} < q_{44} = 54475119544877440894.$$

Furthermore, $a_N := \max\{a_i : i = 0, 1, \ldots, 44\} = a_{17} = 134$. So, from the known properties of continued fractions, we obtain that

$$|t\gamma - k| > \frac{1}{(a_N+2)t}. \tag{4.20}$$

Comparing estimates (4.19)and (4.20), we get right away that

$$\alpha^{k-l} < 7\cdot 136\cdot t < 5\cdot 10^{22}, \tag{4.21}$$

leading to $k - l \leqslant 106$.

Let us now go back to (4.12) to determine an improved upper bound on k.

Put

$$\omega := t\log 2 - k\log\alpha - \log(1-\alpha^{-(k-l)}). \tag{4.22}$$

Therefore, (4.12) implies that

$$|1-e^{\omega}|<\frac{10}{\alpha^{k}}. \tag{4.23}$$

Note that $\omega \neq 0$; thus, we distinguish the following cases. If $\omega>0$ then, from (4.22), we obtain

$$0<\omega\leqslant e^{\omega}-1<\frac{10}{\alpha^{k}}.$$

Replacing ω in the above inequality by its formula (4.22) and dividing both sides of the resulting inequality by $\log\alpha$, we get

$$0<t(\frac{\log 2}{\log\alpha})-k-\frac{\log(1-\alpha^{-(k-l)})}{\log\alpha}<\frac{21}{\alpha^{k}}. \tag{4.24}$$

We now put

$\gamma:=\frac{\log 2}{\log\alpha}$, $\mu:=-\frac{\log(1-\alpha^{-(k-l)})}{\log\alpha}$, $A:=21$ and $B:=\alpha$.

Clearly γ is an irrational number. We also put $N:=5\cdot 10^{19}$, which is an upper bound on t by Lemma 2.1. We therefore apply Lemma 2.1 to inequality (4.24) for all choices $k-l\in\{1,...,106\}$ except when $k-l=1,2,3,6$ and get that

$$k<\frac{\log(Aq/\xi)}{\log B},$$

where $q>6N$ is a denominator of a convergent of the continued fraction of γ such that $\xi=\|\mu q\|-N\|\gamma q\|>0$. Indeed, using Sagemath, we compute

$$q=q_{47}=3233534301552913148 26.$$

We find that if (k,l,t) is a possible solution of equation (1.2) with $\omega>0$ and $k-l\neq 1,2,3,6$, then $k<115$, which is a contradiction with $k>200$.

When $k-l=1,2,3,6$, the parameter μ becomes

$$\mu=\begin{cases} 2 & \text{if} \quad k-l=1; \\ 1 & \text{if} \quad k-l=2; \\ 2-\gamma & \text{if} \quad k-l=3; \\ 3-2\gamma & \text{if} \quad k-l=6. \end{cases}$$

In that case, the corresponding value of ξ from Lemma 2.1 is always negative and therefore the reduction method is not useful for reducing the bound on k in these instances. For this reason we need to use the properties of continued fractions to treat these cases.

For all that, one can see that if $k-l=1,2,3,6$. Then the resulting inequality from (4.24) has the shape

$$0<|a\gamma-b|<\frac{21}{\alpha^k},$$

with γ being an irrational number and $a,\ b\in\mathbb{Z}$. So, one can appeal to the known properties of the convergents of the continued fractions to obtain a nontrivial lower bound for

$$|a\gamma-b|.$$

This clearly gives us an upper bound for k. Let us see. When $k-l=1$, from (4.24), we get that

$$0<t\gamma-(k-2)<\frac{21}{\alpha^k}. \tag{4.25}$$

Let $[a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7,\ldots]=[1,2,3,1,2,3,2,4\ldots]$ be the continued fraction expansion of γ, and let denote p_n/q_n its nth convergent. Recall also that $t<5\cdot10^{19}$ by Lemma 4.1.

Furthermore, $a_N:=\max\{a_i:i=0,1,\ldots,44\}=a_{17}=134$. So, from the known properties of continued fractions, we obtain that

$$|t\gamma-(k-2)|>\frac{1}{(a_N+2)t}. \tag{4.26}$$

Comparing estimates (4.25) and (4.26), we get right away that

$$\alpha^k<21\cdot136\cdot t<2\cdot10^{23}, \tag{4.27}$$

leading to $k<112$.

By the same argument as the one we did before ensures that $k-l<106$ in the case when $k-l=2,3,6$. We omit the details in order to avoid unnecessary repetitions. This completes the analysis of the cases when $k-l=1,2,3,6$. Consequently, $k<119$ always holds.

Suppose now that $\omega<0$. First, note that $\frac{10}{\alpha^k}<\frac{1}{2}$ since $k>200$. Then, from (4.22), we have that

$$|1-e^{\omega}|<\frac{1}{2},$$

thus

$$\frac{1}{2} < e^{\omega} < \frac{3}{2}$$

and therefore

$$e^{|\omega|} < 2.$$

Since $\omega < 0$, we have

$$0 < |\omega| \leqslant e^{|\omega|} - 1 = e^{|\omega|}|e^{-|\omega|} - 1| = e^{|\omega|}|e^{\omega} - 1| < \frac{20}{\alpha^k}.$$

Then we obtain

$$0 < -t\log 2 + k\log\alpha + \log(1 - \alpha^{-(k-l)}) < \frac{20}{\alpha^k}.$$

By the same arguments used for proving (4.22), we obtain

$$0 < k(\frac{\log\alpha}{\log 2}) - t + \frac{\log(1 - \alpha^{-(k-l)})}{\log 2} < \frac{29}{\alpha^k}. \tag{4.28}$$

We now put

$\gamma := \frac{\log\alpha}{\log 2}$, $\mu := \frac{\log(1-\alpha^{-(k-l)})}{\log 2}$, $A := 29$ and $B := \alpha$.

Indeed, with the help of Sagemath, suppose that

$$q = q_{47} = 3689403469796380332 17.$$

We find that if (k, l, t) is a possible solution of the equation (1.2) with $\omega < 0$ and $k - l \neq 1, 2, 3, 6$, then $k < 119$, which is a contradiction with our assumption.

When $k - l = 1, 2, 3, 6$; we have

$$\mu = \begin{cases} -2\gamma & \text{if} \quad k - l = 1; \\ -\gamma & \text{if} \quad k - l = 2; \\ 1 - 2\gamma & \text{if} \quad k - l = 3; \\ 2 - 3\gamma & \text{if} \quad k - l = 6. \end{cases}$$

In these cases, the resolution is done with the properties of continuous fractions as previously, and we will see that $k < 119$ in each case. Thus Theorem 1.3 is proven.

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