The Aligarh Bulletin of Mathematics
Volume 42, Number 2 (2023), 81 - 94
ISSN: 0304-9787

On the solutions of Sobolev type fractional partial differential equations

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(Received: December 7, 2023 Accepted: January 30, 2024)

In memory of Professor Abul Hasan Siddiqi (1943-2020)

Abstract

The purpose of this paper is to study the existence of solutions for the nonlinear Sobolev type fractional partial differential equations with Dirichlet boundary condition. Under suitable assumptions the results are established by using the Leray-Schauder fixed point theorem.

1 Introduction

The existence of solutions for different types of fractional partial differential equations have been studied by many authors [1–3, 22]. These equations are found to be an effective tool to describe certain physical phenomena, such as diffusion processes [14] and viscoelasticity theories [15]. In recent years, increasing interest has been shown by many authors from various fields of science and engineering to study fractional partial differential equations. Some of these fractional equations like one-dimensional time-fractional diffusion-wave equation were used for modeling certain physical processes (see [25]). Regarding fractional partial differential

Keywords and phrases: Fractional partial differential equations, Sobolev type, existence of solutions.

2020 AMS Subject Classification: 26A33, 35R11, 35S10.

equations, Luchko [20] used the Fourier transform method of the variable separation to construct a formal solution and under certain conditions he showed that the formal solution is the generalized solution of the initial-boundary value problem. Further he proved the uniqueness [19] by using the maximum principle for generalized time fractional diffusion equation. By applying the energy inequality, Oussaeif and Bouziani [23] proved the existence and uniqueness of solution for parabolic fractional differential equations in a functional weighted Sobolev space with integral conditions. Parthiban and Balachandran [24] found the solutions of system of fractional partial differential equations by using Adomain decomposition method. Joice Nirmala and Balachandran [16] obtained the solution of time fractional telegraph equation by the same method and analysed the efficiency of the method. Using measure of noncompactness and Monch's fixed point theorem, the existence of solutions is studied by Guo and Zhang [13] for a class of impulsive hyperbolic partial differential equations.

Brill [11] and Showalter [26] investigated the existence problem for semilinear Sobolev type equations in Banach spaces. The Sobolev type semilinear differential equation serves as an abstract formulation of partial differential equations which arise in various applications such as in the flow of fluid through fissured rocks, thermodynamics and shear in second order fluids. Lightbourne and Rankin [21] discussed the Cauchy problem for a partial functional differential equation of Sobolev type. Balachandran et al. [10] established the existence of solutions for Sobolev type semilinear integrodifferential equations whereas Balachandran and Uchiyama [8] studied the same problem for nonlinear integrodifferential equation of Sobolev type in Banach spaces. Several authors have studied the nonlocal Cauchy problem for Sobolev type equations in Banach spaces [5, 6, 9]. Balachandran and Kiruthika [7] discussed the existence problem for abstract fractional integrodifferential equations of Sobolev type. Existence of solutions for fractional integrodifferential equations of Sobolev type with deviating arguments are studied in [17]. In this paper, we extend the method of [22] to discuss the existence problem for fractional order Sobolev type partial differential equations.

2 Preliminaries

In this section, we introduce some notations and basic facts of fractional calculus. Let $\Omega \subset \mathbb{R}$ and $C(J, \mathbb{R})$ is the Banach space of all continuous functions from $J = [0, T]$ into $\mathbb{R}$. Let $\Gamma(\cdot)$ denote the gamma function. For any positive number $0 < \alpha < 1$, the Riemann Liouville derivative and Caputo derivative are defined as follows:

Definition 2.1. *[18] The Riemann-Liouville partial fractional integral operator of order $\alpha > 0$ with respect to t of a function $z(x,t)$ is defined by*

$$I^{\alpha} z(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(x,s)\, \mathrm{d}s.$$

Definition 2.2. *[18] The Riemann-Liouville partial fractional derivative of order $\alpha > 0$ of a function $z(x,t)$ with respect to t of the form*

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} z(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{z(x,s)}{(t-s)^{\alpha}}\, \mathrm{d}s.$$

Definition 2.3. *[18] The Caputo partial fractional derivative of order $\alpha > 0$ with respect to t of a function $z(x,t)$ is defined as*

$$\frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}} z(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^{\alpha}} \frac{\partial z(x,s)}{\partial s}\, \mathrm{d}s.$$

There has been a significant development in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives in the past few years [4,12]. The Riemann Liouville and Caputo fractional derivatives are linked by the following relationship.

$$\frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}} z(x,t) = \frac{\partial^{\alpha}}{\partial t^{\alpha}} z(x,t) - \frac{z(x,0)}{\Gamma(1-\alpha) t^{\alpha}}.$$

In this paper, we consider the Sobolev type fractional partial differential equation of the form

$$\frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}} [u(x,t) - \Delta u(x,t)] = \Delta u(x,t) + f(t, u(x,t)), \quad t \in J, \tag{2.1}$$

where $0 < \alpha < 1$, Ω is a bounded subset of $\mathbb{R}$ with smooth boundary $\partial\Omega$, $J = [0,T]$ and $f : J \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function. The initial and boundary conditions are given by

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{2.2}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times J. \tag{2.3}$$

where $u_0(x) \in C^2(\Omega)$. In order to establish the main result we assume the following conditions:

(H_1) $f(t,u)$ is continuous with respect to u, Lebesgue measurable with respect to t and satisfies

$$\frac{1}{\int_\Omega \phi(x)\,\mathrm{d}x}\int_\Omega \phi(x)f(t,u)\,\mathrm{d}x \le f\big(t, \int_\Omega \phi(x)u(x,t)\,\mathrm{d}x / \int_\Omega \phi(x)\,\mathrm{d}x\big),$$

where $\phi(x)$ is an eigenfunction.

(H_2) There exists an integrable function $m(t) : J \to [0,\infty)$ and a constant $L > 0$ such that

$$\begin{aligned} \|f(t,u)\| &\le m(t)\|u\| \\ \text{and} \int_0^T (T-s)^{\alpha-1} m(s)\mathrm{d}s &\le L \text{ for some } \alpha > 0. \end{aligned}$$

It is easy to show that the initial value problem (2.1) is equivalent to the following equation

$$\begin{aligned} u(x,t) &= u_0(x) - u_0''(x) + \Delta u(x,t) \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\Delta u(x,s) + f(s,u(x,s))]\,\mathrm{d}s. \end{aligned} \tag{2.4}$$

3 Existence Result

Consider the following eigenvalue problem

$$\left.\begin{aligned} \Delta u + \lambda u &= 0, \quad (x,t) \in \Omega \times J, \\ u &= 0, \quad (x,t) \in \partial\Omega \times J, \end{aligned}\right\} \tag{3.1}$$

where λ is a constant not depending on the variables x and t. The theory of eigenvalue problems is well documented in [27]. Thus, for $x \in \Omega$ the smallest eigenvalue λ_1 of the problem (3.1) is positive and the corresponding eigenfunction

$\phi(x) \geq 0$. Now we define the function $U(t)$ as

$$U(t) = \frac{\int\limits_{\Omega} u(x,t)\phi(x)\,\mathrm{dx}}{\int\limits_{\Omega} \phi(x)\,\mathrm{dx}}. \tag{3.2}$$

Theorem 3.1. *Assume that (H1)-(H2) holds and suppose that*

$$(\lambda_1 T^{\alpha} + L\alpha) < (1+\lambda_1)\Gamma(\alpha+1). \tag{3.3}$$

Then there exists at least one solution for the initial value problem (2.1) on J.

Proof. First we have to prove that the initial value problem (2.1) has a solution if and only if the equation

$$U(t) = U(0) - V(0) - \lambda_1 U(t) - \frac{\lambda_1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} U(s)\,\mathrm{ds}$$
$$+ \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} f(s, U(s))\,\mathrm{ds},$$

or

$$U(t) = \frac{1}{1+\lambda_1}[U^*(0) - \frac{\lambda_1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} U(s)\,\mathrm{ds}$$
$$+ \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} f(s, U(s))\,\mathrm{ds}], \tag{3.4}$$

where $U^*(0) = U(0) - V(0)$ and $V(0) = \Big(\int\limits_{\Omega} u_0''(x)\phi(x)\,\mathrm{dx}\Big) / \int\limits_{\Omega} \phi(x)\,\mathrm{dx}$, has a solution.

Step 1. The proof of sufficiency is similar to that of Lemma 3.1 [22]. To prove the necessary part, let $u(x,t)$ be a solution of (2.1). This implies $u(x,t)$ is a solution of (2.4). Now multiplying both sides of equation (2.4) by $\phi(x)$ and integrating with respect to $x \in \Omega$, we get

$$\int\limits_{\Omega} \phi(x)u(x,t)\,\mathrm{dx} = \int\limits_{\Omega} \phi(x)u_0(x)\,\mathrm{dx} - \int\limits_{\Omega} \phi(x)u_0''(x)\,\mathrm{dx} + \int\limits_{\Omega} \phi(x)\Delta u(x,t)\,\mathrm{dx}$$
$$+ \frac{1}{\Gamma(\alpha)} \int\limits_{\Omega} \phi(x) \int_0^t (t-s)^{\alpha-1}\Delta u(x,s)\,\mathrm{ds\,dx}$$
$$+ \frac{1}{\Gamma(\alpha)} \int\limits_{\Omega} \phi(x) \int_0^t (t-s)^{\alpha-1} f(s,u(x,s))\,\mathrm{ds\,dx}.$$

Using Green's formula and assumption (H_1), we get

$$\begin{aligned} U(t) \quad &\leq \quad \frac{1}{1+\lambda_1}[U(0) - V(0) - \frac{\lambda_1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} U(s)\,\mathrm{ds} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} f(s,U(s))\,\mathrm{ds}]. \end{aligned} \tag{3.5}$$

Choose $b \geq \dfrac{\|U(0)\|(\lambda_1 T^\alpha + L\alpha)}{(1+\lambda_1)\Gamma(\alpha+1) - (\lambda_1 T^\alpha + L\alpha)}$ and let $K = \{U : U \in C(J,\mathbb{R}),$ $\| U(t) - U(0) \| \leq b\}$. Define the nonlinear operator

$$F : C(J,\mathbb{R}) \to C(J,\mathbb{R})$$

as

$$\begin{aligned} FU(t) \quad &= \quad \frac{1}{1+\lambda_1}[U(0) - V(0) - \frac{\lambda_1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} U(s)\,\mathrm{ds} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} f(s,U(s))\,\mathrm{ds}]. \end{aligned} \tag{3.6}$$

Clearly $U(0) \in K$. This means that K is nonempty. From our construction of K, we can say that K is closed and bounded. Now for any $U_1, U_2 \in K$ and for any $a_1, a_2 \geq 0$ such that $a_1 + a_2 = 1$,

$$\| a_1U_1(t) + a_2U_2(t) - U(0) \| \leq a_1 \| U_1(t) - U(0) \| + a_2 \| U_2(t) - U(0) \|$$

$$\leq a_1 b + a_2 b = b.$$

Thus $a_1U_1 + a_2U_2 \in K$. Therefore K is a closed bounded convex set. Next we have to prove that the operator F maps K into itself.

$$\begin{aligned}\| FU(t) - FU(0) \| &= \frac{1}{1+\lambda_1} \Bigg\| \frac{\lambda_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(s)\, \mathrm{ds} + \\ &\quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, U(s))\, \mathrm{ds} \Bigg\| \\ &\leq \frac{\lambda_1}{(1+\lambda_1)\Gamma(\alpha)} (\|U(0)\| + b) \int_0^t (t-s)^{\alpha-1} \mathrm{d}s \\ &\quad + \frac{1}{(1+\lambda_1)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, U(s))\|\, \mathrm{ds}.\end{aligned}$$

Then by using (H_2), we get

$$\begin{aligned}\| FU(t) - FU(0) \| &\leq \frac{\lambda_1}{(1+\lambda_1)\Gamma(\alpha)} (\|U(0)\| + b) \left(\int_0^t (t-s)^{\alpha-1} \mathrm{d}s \right) \\ &\quad + \frac{1}{(1+\lambda_1)\Gamma(\alpha)} \int_0^t m(s)(t-s)^{\alpha-1} \|U(s)\|\, \mathrm{d}s \\ &\leq \frac{\lambda_1}{(1+\lambda_1)\Gamma(\alpha)} (\|U(0)\| + b) \int_0^T (T-s)^{\alpha-1} \mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(1+\lambda_1)\Gamma(\alpha)}(\|U(0)\|+b)\int_0^T m(s)(T-s)^{\alpha-1}\mathrm{d}s\\
&\le \frac{\lambda_1 T^\alpha}{(1+\lambda_1)\alpha\Gamma(\alpha)}(\|U(0)\|+b)+\frac{L}{(1+\lambda_1)\Gamma(\alpha)}(\|U(0)\|+b)\\
&=\frac{(\|U(0)\|+b)}{(1+\lambda_1)\Gamma(\alpha+1)}[\lambda_1 T^\alpha+L\alpha]\\
&\le b.
\end{aligned}
$$

Therefore F maps K into itself. Now define a sequence $\{U_k(t)\}$ in K such that

$$U_0(t)=U(0)-V(0) \text{ and } U_{k+1}(t)=U_k(t), \quad k=0,1,2,\ldots$$

Since K is closed, there exists a subsequence $\{U_{k_i}(t)\}$ of $U_k(t)$ and $\widetilde{U}(t)\in K$ such that

$$\lim_{k_i\to\infty} U_{k_i}(t)=\widetilde{U}(t).$$

Then Lebesgue's dominated convergence theorem yields that

$$
\begin{aligned}
\widetilde{U}(t)=\widetilde{U}(0)-\widetilde{V}(0)-\frac{\lambda_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\widetilde{U}(s)\,\mathrm{d}s\\
+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\widetilde{U}(s))\mathrm{d}s.
\end{aligned}
$$

Next we claim that F is completely continuous.

Step 2. For that first we prove $F:K\to K$ is continuous. Let $\{U_m(t)\}$ be a convergent sequence in K such that $U_m(t)\to U(t)$ as $m\to\infty$. Then for any $\epsilon>0$, let

$$\|U_m(t)-U(t)\|\le \frac{(1+\lambda_1)\Gamma(\alpha+1)}{2\lambda_1 T^\alpha}\epsilon.$$

By assumption (H_1),

$$f(t,U_m(t))\longrightarrow f(t,U(t)),$$

for each $t \in J$ and since

$$\|f(t, U_m(t)) - f(t, U(t))\| \leq \frac{(1+\lambda_1)\Gamma(\alpha+1)}{2T^\alpha}\epsilon,$$

we have

$$\begin{aligned}
&\|FU_m(t) - FU(t)\| \\
&\leq \frac{\lambda_1 T^\alpha}{(1+\lambda_1)\Gamma(\alpha+1)}\|U_m(t) - U(t)\| \\
&+ \frac{1}{(1+\lambda_1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s, U_m(s)) - f(s, U(s))\|ds \\
&\leq \frac{\epsilon}{2} + \frac{1}{(1+\lambda_1)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|f(s, U_m(s)) - f(s, U(s))\|ds \\
&\leq \epsilon.
\end{aligned}$$

Thus $FU_m(t) \to FU(t)$ as $m \to \infty$ and so F is continuous.

Moreover, for $U \in K$,

$$\begin{aligned}
\| FU(t) \| &\leq \|FU(t) - FU(0)\| + \|FU(0)\| \leq b + \frac{1}{1+\lambda_1}[\|U(0) + \|V(0)\|] \\
&\leq \|U(0)\| + \|V(0)\| + b.
\end{aligned}$$

Hence FK is uniformly bounded. Now it remains to show that F maps K into an equicontinuous family.

Step 3. Now let $U \in K$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \leq T$, by the assumptions $(H1) - (H2)$ we obtain

$$\begin{aligned}
&\| FU(t_1) - FU(t_2) \| \\
&\leq \frac{\lambda_1}{(1+\lambda_1)\Gamma(\alpha)}(\|U(0)\| + b)\int_0^{t_1}\left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right)\mathrm{d}s \\
&+ \frac{\lambda_1}{(1+\lambda_1)\Gamma(\alpha)}(\|U(0)\| + b)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathrm{d}s
\end{aligned}$$

$$+ \frac{1}{(1+\lambda_1)\Gamma(\alpha)} \left\| \int_0^{t_1} \Big((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \Big) f(s, U(s))\, \mathrm{d}s \right\|$$

$$+ \frac{1}{(1+\lambda_1)\Gamma(\alpha)} \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, U(s))\, \mathrm{d}s \right\|$$

$$\leq \frac{\lambda_1}{(1+\lambda_1)\Gamma(\alpha)} \left(\|U(0)\| + b \right) \int_0^{t_1} \left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right) \mathrm{d}s$$

$$+ \frac{\lambda_1}{(1+\lambda_1)\Gamma(\alpha)} \left(\|U(0)\| + b \right) \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathrm{d}s$$

$$+ \frac{L(\|U(0)\| + b)}{(1+\lambda_1)\Gamma(\alpha)} \left\| \int_0^{t_1} \Big((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \Big) \mathrm{d}s \right\|$$

$$+ \frac{L(\|U(0)\| + b)}{(1+\lambda_1)\Gamma(\alpha)} \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathrm{d}s \right\|$$

The right hand side of the above inequality is independent of $U \in K$ and goes to zero as $t_1 \to t_2$. Thus, F maps K into an equicontinuous family of functions. In the view of the Ascoli-Arzela theorem, F is completely continuous. Then by the Leray-Schauder fixed point theorem, F has a fixed point in K, which is a solution of (2.1). □

4 Abstract Fractional Sobolev Equation

The fractional partial differential equation (2.1) can be discussed in abstract setting as in [7, 11, 21] and it can be written as

$$\begin{aligned} \frac{{}^C\partial^\alpha}{\partial t^\alpha}[Bu(x,t)] &= Au(x,t) + f(t, u(x,t)), \quad t \in J, \\ u(x,0) &= u_0(x) \end{aligned} \tag{4.1}$$

where $B = I - A$ and $A = \Delta$. Suppressing the phase variable and considering in general Banach space X the above equation can be written as abstract fractional differential equation of the form

$$\begin{aligned} {}^{C}D^{\alpha}[Bu(t)] &= Au(t) + f(t, u(t)), t \in J, \\ u(0) &= u_0 \end{aligned} \tag{4.2}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative with $0 < \alpha < 1$. The operators A and B are linear with domains contained in a Banach space X and ranges contained in a Banach Space Y and the operators $A : D(A) \subset X \to Y$ and $B : D(B) \subset X \to Y$ satisfy the following hypotheses:

(C1) A and B are closed linear operators,

(C2) $D(B) \subset D(A)$ and B is bijective,

(C3) $B^{-1} : Y \to D(B)$ is compact,

(C4) $B^{-1}A : X \to D(B)$ is continuous.

The nonlinear operator $f : J \times X \to Y$ is continuous. It is easy to prove that the equation (4.2) is equivalent to the integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B^{-1} Au(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B^{-1} f(s, u(s)) ds. \tag{4.3}$$

The existence problem for the equation (4.3) has been already discussed in [7, 17].

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