The Aligarh Bulletin of Mathematics
Volume 42, Numbers 2 (2023), 59 - 67

ISSN: 0304-9787

Similarity and equivalence in poloid

Hasan Keleş
Karadeniz Technical University
Department of Mathematics, Trabzon, Türkiye
Email: hasankeles@ktu.edu.tr

(Received: November 11, 2022 Accepted: December 31, 2023)

Abstract

This study is about equivalence and similarity of regular matrices. There is no square or regularity requirement for equivalent matrices. The condition of regularity of matrices is sought since matrix division is used in the study. Shortly, theorems, lemmas and examples are given on equivalence and similarity on regular matrices in the study.

1 Introduction

Arthur Cayley studied and researched the matrix in depth about 1850. The matrix subject started to be taught first in graduate courses and then in undergraduate courses from the 19th century [1]. The subject similarity is the ideal application for solving the problem of simultaneous diagonalization of multiple matrices [2]. Tadej Starčič made another contribution to this topic in 2022 with his paper "Isotropy groups of the action of orthogonal similarity on symmetric matrices" [3]. Jiang,Cheng and Ling have obtained important results on Similarity of Quaternion Matrices and Applications [4].

Fuhrmann described the projection map on the require matrix in [5]. The set of all matrices of order n over a field $\mathbb{F}$ is denoted by $\mathbb{M}_m^n(\mathbb{F})$. The transpose of $A \in \mathbb{M}_m^n(\mathbb{F})$ is denoted by A^T. The set of all regular matrices of order n over a

Keywords and phrases : Similarity, equivalence, relation, equivalence relation, matrix theory, division of matrices
2010 AMS Subject Classification : 76M55, 03D50, 03E20, 37A20, 15A09, 11R52

field $\mathbb{F}$ is denoted by $\mathbb{M}_n(\mathbb{F})$ in [6, 7].

Theorem 1.1. *(See Keleş [8]) Let $A, B \in \mathbb{M}_n(\mathbb{F})$. Then, the solution of the linear matrix equation $XA = B$ is*

$$X = \left(\frac{B^T}{A^T}\right)^T. \tag{1.1}$$

Proof. The solution of the equation $AX = B$ is $X = \dfrac{B}{A}$ for all $A, B, X \in \mathbb{M}_n(\mathbb{F})$.

$$XA = B \iff (XA)^T = B^T \iff A^T X^T = B^T. \tag{1.2}$$

$$X^T = \frac{1}{|A^T|}\left[\begin{pmatrix} B^T A^T \\ ij \end{pmatrix}_{ij}\right] \implies X = \frac{1}{|A^T|}\left[\begin{pmatrix} B^T A^T \\ ij \end{pmatrix}_{ji}\right]^T. \tag{1.3}$$

$$X = \frac{1}{|A^T|}\left[\begin{pmatrix} B^T A^T \\ ij \end{pmatrix}_{ij}\right] = \left(\frac{B^T}{A^T}\right)^T. \tag{1.4}$$

□

Definition 1.1. *Let $A, B \in \mathbb{M}_m^n(\mathbb{F})$ be any two matrices. We called matrix A is equivalent to matrix B, if there are $Q \in \mathbb{M}_m(\mathbb{F})$ and $P \in \mathbb{M}_n(\mathbb{F})$ such that*

$$B = QBP.$$

It is denoted by $B \equiv A$.

$B \equiv A$ in matrices is equivalence relation, i.e.,

(i) $A \equiv A$,

(ii) If $A \equiv B$, then $B \equiv A$.

(iii) If $A \equiv B, B \equiv C$ then $A \equiv C$.

These properties are often expressed by saying that the similarity relation $\equiv$ is an equivalence relation on the set of $n \times n$ matrices in [7, 9, 10, 11]. Here is an example showing how these properties are used.

The division operation in matrices is also an equivalence relation. This is easy to show.

(i) If $A \mid A$ for all $A \in M_n(\mathbb{F})$.

(ii) If $A \mid B$, then $B \mid A$ for any two $A, B \in \mathbb{M}_n(\mathbb{F})$ in [5, 8].

(iii) If $A \mid B$ and $B \mid C$, then $A \mid C$ for any three $A, B, C \in \mathbb{M}_n(\mathbb{F})$.

Lemma 1.1. *For all $A, B \in M_n(\mathbb{F})$, there are $P, Q \in \mathbb{M}_n(\mathbb{F})$, such that $B = QAP$.*

Proof. For all $A, B \in \mathbb{M}_n(\mathbb{F})$

$$A \mid B \Leftrightarrow \textit{there exist } P \in \mathbb{M}_n(\mathbb{F}) \textit{ such that } B = AP.$$

If $P \neq \dfrac{B}{A}$ is selected, then

$$B \neq AP.$$

There exist $Q \in \mathbb{M}_n(\mathbb{F})$ such that

$$B = QAP.$$

By Theorem 1.1

$$Q = \left(\frac{\left(BP^{-1}\right)^T}{A^T} \right)^T$$

$$B = QAP.$$

□

The following corollary is given for convenience in applications.

Corollary 1.1. *(i) The solution of the equation $XA = B$ is used. The matrix Q is obtained.*

(ii) This proof is also made by utilizing the solution of the equation $AX = B$. In this case, the P matrix is calculated.

(iii) Any two regular matrices of the same order are equivalent.

Definition 1.2. *Two matrices $A, B \in \mathbb{M}_n(\mathbb{F})$ are said to be similiar or conjugated if they are the same linear transformation in possibly different bases,*

$$A = PBP^{-1}$$

where P is an invertible change of basis matrix. If A similiar to B, then it is denote as $A \sim B$ in [5, 7, 9]

If $A \sim B$, then $B \sim A$. Because, if $A \sim B$, then there exist $P \in \mathbb{M}_n(F)$

$$A = PBP^{-1}.$$

$$AP = PBP^{-1}P$$
$$P^{-1}AP = P^{-1}PB$$
$$P^{-1}AP = B$$

Similarity is an equivalence relation. This is easy to show in [5].

(i) If $A \sim A$ for all $A \in \mathbb{M}_n(\mathbb{F})$.

(ii) If $A \sim B$, then $B \sim A$ for any two $A, B \in \mathbb{M}_n(\mathbb{F})$.

(iii) If $A \sim B$ and $B \sim C$, then $A \sim C$ for any three $A, B, C \in \mathbb{M}_n(\mathbb{F})$..

Lemma 1.2. *For all $B \in \mathbb{M}_n(\mathbb{F})$, there are $P \in \mathbb{M}_n(\mathbb{F})$, such that $B = P^{-1}XP$, where $X \in \mathbb{M}_n(\mathbb{F})$.*

Proof. For all $A, B \in \mathbb{M}_n(\mathbb{F})$

$$A \mid B \Leftrightarrow \textit{there exist } P \in \mathbb{M}_n(\mathbb{F}) \textit{such that } B = AP.$$

If $P^{-1} \mid A$, then

$$A = P^{-1}X, \textit{where } X \in \mathbb{M}_n(\mathbb{F}).$$

And

$$B = P^{-1}XP.$$
$$X = \frac{BP^{-1}}{P^{-1}}, \textit{where } X \in \mathbb{M}_n(\mathbb{F}).$$

□

The following corollary is given for convenience in applications.

Corollary 1.2. *(i) The equation $AX = B$ is used for the solution of the equation $B = P^{-1}XP$.*

(ii) This proof is also made by utilizing the solution of the equation $XA = B$. In this case, the matrix X is calculated.

(iii) Every regular matrix is similar to at least one regular matrix of the same order.

Definition 1.3. *A group is a set $\mathbb{P}$ equipped with a binary operation $* : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ that associates an element $a * b \in \mathbb{P}$ to every pair of elements $a, b \in \mathbb{P}$, and having the following properties: $*$ is associative, has an identity element $e \in \mathbb{P}$, and every element in $\mathbb{P}$ is invertible. More explicitly, this means that the following equations hold for all $a, b, c, d \in \mathbb{P}$:*
(P1) $a * (b * c) = (a * b) * c$, (associativity);
(P2) $a * e = e * a = a$, (identity);
(P3) For every $a \in \mathbb{P}$, there is some $a^{-1} \in \mathbb{P}$ such that $a * a^{-1} = a^{-1} * a = e$. (inverse);
(P4) For every $a \in \mathbb{P}\backslash\{e\}$, there are some $d, f \in \mathbb{P}\backslash\{e\}$ such that $b*f = f*d = a$ with $b \neq d$. (escort).

A set $\mathbb{P}$ together with an operation $* : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ and an element e satisfying only conditions (P1), (P2), (P3) and (P4) is called a poloid. It is denoted by $(\mathbb{P}, *)$ in [8].

2 Similarity and Equivalence in Poloid

Let us start with the new definition of equivalence in poloid. The concept of equivalence is evolved with this definition.

Definition 2.1. *Two matrices $a, b \in \mathbb{P}$ are said to be equivalent if there are $p, q \in \mathbb{P}$ sucf that,*

$$a = q * b * p.$$

If a equivalence to b then it is denote as $a \equiv b$.

Lemma 2.1. *Let $(\mathbb{P}, *)$ be a poloid. If any two elements $a, b \in \mathbb{P}$ then $a \equiv b$.*

Proof. Let $(\mathbb{P}, *)$ be a poloid. For any two $a, b \in \mathbb{P}$, $b = a * p$. If $p \neq a^{-1} * b$ is selected, then there exist $q \in \mathbb{P}$, such that $b = q * a * p$.

$$a \equiv b.$$

□

Example 2.1. *If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix}$, then $P = \dfrac{B}{A} = \begin{bmatrix} 3 & -10 \\ -2 & 7 \end{bmatrix}$. If*

P is selected such that $P = \dfrac{B}{A} = \begin{bmatrix} 1 & -10 \\ -2 & 3 \end{bmatrix}$ *then*

$$AP \neq B.$$

$$QAP = B \Rightarrow Q = \left(\frac{B^T}{(AP)^T}\right)^T = \begin{bmatrix} -\frac{3}{17} & \frac{4}{17} \\ -\frac{16}{17} & \frac{27}{17} \end{bmatrix}$$

$$QAP = \begin{bmatrix} -\frac{3}{17} & \frac{4}{17} \\ -\frac{16}{17} & \frac{27}{17} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -10 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix} = B.$$

$$A \equiv B.$$

The definition of similarity of the following two matrices to selected a matrix.

Definition 2.2. *Let $\mathbb{P}$ be a poloid. The elements $a, b \in \mathbb{P}$ are said to be similar to selected a element $q \in \mathbb{P}$ if $p := a^{-1} * q, q * p^{-1} = a$ and $p^{-1} * q = b$. It is denoted by $a_q \sim b_q$.*

Lemma 2.2. *Let $(\mathbb{P}, *)$ be a poloid. The following are equivalent for two elements $a, b \in \mathbb{P}$ of the poloid.*

(i) If $q \in \mathbb{P}$ is selected, then $a_q \sim b_q$.

(ii) $a \sim b$.

Proof. Let $(\mathbb{P}, *)$ be a poloid. For any $a, b, p \in \mathbb{P}, a * p = p * b = q$, where $a = b$ by (P4). If $q \in \mathbb{P}$ is selected

$$p = a^{-1} * q, q * p^{-1} = a \textit{ and } p^{-1} * q = b.$$

(i) Then

$$a_q \sim b_q.$$

(ii) If $a_q \sim b_q$, then

$$p = a^{-1} * q, q * p^{-1} = a \textit{ and } p^{-1} * q = b.$$

$$p^{-1} * a * p = b.$$

$$a \sim b.$$

□

Example 2.2. *If two matrices* $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix}$ *and for selected the matrix* $Q = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, *then*

$$P = \frac{Q}{A} = \begin{bmatrix} 5 & -1 \\ -3 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

$$QP^{-1} = Q = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = A$$

$$P^{-1}Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix} = B$$

$$A_Q \sim B_Q,$$

and

$$P^{-1}AP = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix} = B.$$

Then

$$A \sim B.$$

3 Conclusions and Discussions

Some minor changes are needed in the definitions of equivalence and similarity on the poloid structure in this study. These minor changes operation of division in matrices. In addition, the definition of poloid made definitional expressions even more meaningful. There are many research problems in this field, since not much study has been done on the poloid structure.

4 Acknowledgments

Endless thanks to Arthur Cayley for her contributions to science. I would like to express my gratitude and gratefulness to each of the reviewers for their comments and suggestions to my article.

References

[1] Avital, S., *History of Mathematics Can Help Improve Instruction and Learning*, Mathematical Association of America, 3-12, 1995.

[2] Evard, J. C. and Gracia, J. M., *On Similarities of Class C^P and Applications to Matrix Differential Equations*, Linear Algebra and its Applications,(137–138), 363-386, 1990.

[3] Starčič, T., *Isotropy groups of the action of orthogonal similarity on symmetric matrices*, Linear and Multilinear Algebra, Taylor and Francis Group, 2022. https://doi.org/10.1080/03081087.2022.2044746.

[4] Jiang,T. Cheng, X. and Ling,S., *An Algebraic Relation between Consimilarity and Similarity of Quaternion Matrices and Applications*, Journal of Applied Mathematics, Volume 2014, Article ID 795203, 5 pages, 2014. http://dx.doi.org/10.1155/2014/795203.

[5] Paul, A. F., *On Strict System Equivalence and Similarity*, International Journal of Control, 25(1), 5-10, 1977.

[6] Prokip V. M., *On the Similarity of Matrices AB and BA Over a field*, Carpathian Math. Publ. 10(2), 352–359, 2018. doi:10.15330/cmp.10.2.352-359.

[7] Nicholson W. K., *Linear Algebra with Applications*, Creative Commons License,University of Calgary, 2021.

[8] Keleş, H., *Poloid and Matrices*, The Aligarh Bulletin of Mathematics, 40(1), 2022.

[9] Maxim, N., *Notes on Linear Algebra and Matrix Analysis*, Version 0.1.1, 4(3), 2006.

[10] Keleş, H., *Lineer Cebire Giriş-I-*, Bordo Puplication, Trabzon, Turkiye, 2015.

[11] Duane, P. A., *Solvability of the Matric Equation $AX = B$*, Linear Algebra and Its Applzcatzons 13 (1976). 177-164.