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Elzaki decomposition method for solving duffing equation

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Abstract

In this study, we have employed Elzaki Adomian Decomposition Method (EADM) to solve Duffing Equation. This method is depends on Elzaki Transform and Adomian decomposition method. Besides, three examples are represented to illustrate the validity and accuracy of the proposed method, as shown in the figures.

1 Introduction

Several natural systems are modeled by nonlinear differential equations which cannot be easily solved. Therefore, the investigation of solving such equations by other methods is an important topic of the research. Lots of different methods have been developed to get exact and approximate solutions to these equations in recent years. Some of these methods are the Adomian decomposition method, Differential transformation method, Homotopy perturbation method, Tanh method, Elzaki Transform, and Variational iteration method [2,4–13]. Also, these equations have been solved by Laplace Decomposition Method, and Fourier decomposition method [1, 3, 14].

Keywords and phrases: Elzaki Transform, Duffing equation, Elzaki Decomposition Method.
2020 AMS Subject Classification: 34A34

In this study, approximate solution of Duffing equation has been found by using Elzaki decomposition method. The Duffing equation is an ordinary differential equation of second order, that is

$$y'' + \alpha y' + \beta y + \gamma y^3 = h(x) \tag{1.1}$$

$$y(0) = A, y'(0) = B, \tag{1.2}$$

where $\alpha, \beta, \gamma, A, B$ are real constants.
In this article, three examples that were previously solved by other methods were solved using EADM. The results have been seen consistent with the literature.

2 Preliminaries

Definition 2.1. *The Elzaki transform of* $h(x)$ *is given by*

$$E(h(x)) = u \int_0^\infty e^{-\frac{x}{u}}.h(x)dx, x > 0.$$

Theorem 2.1. $[4, 13]$ *The Elzaki Transformations of some functions :*

$h(x)$	$E(h(x))$
1	u^2
x^n	$n!u^{n+2}$
e^{ax}	$\frac{u^2}{1-au}$
$cosax$	$\frac{u^2}{1+a^2u^2}$
sin ax	$\frac{au^3}{1+a^2u^2}$.

Theorem 2.2. $[4, 13]$*If* $E(h(x)) = A(u)$*, then*

$$\begin{aligned} i)E\left[h'(x)\right] &= \frac{A(u)}{u} - uh(0) \\ ii)E\left[h''(x)\right] &= \frac{A(u)}{u^2} - h(0) - uh'(0). \end{aligned}$$

3 EADM for the Solution of Duffing Equation

Consider the Duffing equation (1.1) with ICs (1.2) and by applying Elzaki transform to Eq (1.1), we get

$$E(y'') + \alpha E(y') + \beta E(y) + \gamma E(y^3) = E\left(h(x)\right)$$

$$\frac{T}{u^2} - y(0) - uy'(0) + \alpha\left(\frac{T}{u} - uy(0)\right) + \beta T + \gamma E(y^3) = E\left(h(x)\right)$$

$$\frac{T}{u^2} = E(f(x)) + y(0) + uy'(0) - \alpha\Big(\frac{T}{u} - uy(0)\Big) - \beta T - \gamma E(y^3)$$

$$T = u^2y(0) + u^3y'(0) - (\alpha u + \beta u^2)T + \alpha u^3 y(0) - \gamma u^2 E(y^3) + u^2 E(h(x)) \quad (3.1)$$

apply the inverse Elzaki transform to Eq (3.1), we obtain :

$$E^{-1}\left(T\right) = E^{-1}\Big[u^2y(0) + u^3y'(0) - \left(\alpha u + \beta u^2\right)T + \alpha u^3y(0) - \gamma u^2E(y^3) + u^2E\left(h(x)\right)\Big]$$

So we get the following iteration relation.

$$y_{n+1} = E^{-1}\left[-\left(\alpha u + \beta u^2\right)E\left(y_n\right)\right] - \gamma E^{-1}\left[u^2E(A_n)\right],$$

where A_n's Adomian polynomials

$$A_0 = y_0^3,\ A_1 = 3y_1.(y_0)^2, A_2 = 3y_2.(y_0)^2 + 3y_0(y_1)^2$$
$$A_3 = 3y_3(y_0)^2 + 6y_0y_1.y_2 + (y_1)^3$$

and

$$y_0 = E^{-1}\left[u^2y(0) + u^3y'(0) + \alpha u^3y(0) + u^2E\left(h(x)\right)\right]$$

we can use the taylor expansion of the function h at $x = 0$, is given by

$$h(x) \approx \sum_{i=0}^{K} a_i x^i$$

than we get approximation solution as

$$y \approx \sum_{i=0}^{K} y_i.$$

4 Numerical Examples

Example 4.1. $[1, 8, 10, 11]$*Consider the following Duffing equation :*

$\alpha = 0, \beta = 3, \gamma = -2$

$$y'' + 3y - 2y^3 = cosx. \sin 2x \tag{4.1}$$

with ICs

$$y(0) = 0, y'(0) = 1$$

The exact solution is

$$y(x) = \sin x$$

$$h(x) = cosx.sin2x \approx 2x - \frac{7x^3}{3} + \frac{61x^5}{60} - \frac{547x^7}{2520} + \frac{703x^9}{25920}.$$

$$\begin{aligned} y_0 &= E^{-1}\left[u^3 + u^2\left(2u^3 - 14u^5 + 122u^7 - 1094u^9\right)\right] \\ &\approx x + \frac{x^3}{3} - \frac{7}{60}x^5 + \frac{61x^7}{2520} - \frac{547}{181440}x^9 \end{aligned}$$

$$\begin{aligned} y_1 &= E^{-1}\left[-3u^2 E\left(y_0\right)\right] + 2E^{-1}\left[u^2 E(A_0)\right] \\ &= E^{-1}\left[-3u^2\left(u^3 + u^2\left(2u^3 - 14u^5 + 122u^7\right)\right) + 2E^{-1}\left(u^2 E(-\frac{1}{60}x^7 + x^5 + x^3)\right)\right] \\ &= E^{-1}\left[-3u^2\left(u^3 + u^2\left(2u^3 - 14u^5 + 122u^7\right)\right) + 2E^{-1}\left(u^2\left(-84u^9 + 120u^7 + 6u^5\right)\right)\right] \\ &= E^{-1}\left[-3u^5 + 6u^7 + 282u^9 - 534u^{11}\right] \\ &\approx -\frac{x^3}{2} + \frac{x^5}{20} + \frac{47}{840}x^7 - \frac{89}{60480}x^9 \end{aligned}$$

$$\begin{aligned}
A_1 &= 3y_1.(y_0)^2\\
&= 3\left(-\frac{x^3}{2}+\frac{x^5}{20}+\frac{47}{840}x^7-\frac{89x^9}{60480}\right)\left(x+\frac{x^3}{3}-\frac{7}{60}x^5+\frac{61x^7}{2520}-\frac{547x^9}{181440}\right)^2\\
&\approx -\frac{3x^5}{2}-\frac{17x^7}{20}
\end{aligned}$$

$$\begin{aligned}
y_2 &= E^{-1}\left[-3u^2E\left(y_1\right)\right]+2E^{-1}\left[u^2E(A_1)\right]\\
&= E^{-1}\left[-3u^2\left(-3u^5+6u^7+282u^9-534u^{11}\right)\right]+2E^{-1}\left[-180u^9-4284u^{11}\right]\\
&= E^{-1}\left[9u^7-378u^9-9414u^{11}\right]\\
&= \frac{3x^5}{40}-\frac{3x^7}{40}-\frac{523}{20160}x^9
\end{aligned}$$

$$A_2 = 3y_2.(y_0)^2+3y_0\left(y_1\right)^2 \approx \frac{39x^7}{40}$$

$$\begin{aligned}
y_3 &= E^{-1}\left[-3u^2E\left(y_2\right)\right]+2E^{-1}\left[u^2E(A_2)\right]\\
&= E^{-1}\left[-27u^9+10962u^{11}\right]\\
&= -\frac{3x^7}{560}+\frac{29}{960}x^9
\end{aligned}$$

$$A_3 = 3y_3.(y_0)^2+6y_0y_1y_2+\left(y_1\right)^3 \approx \frac{-207x^9}{560}$$

$$\begin{aligned}
y_4 &= E^{-1}\left[-3u^2E\left(y_3\right)\right]+2E^{-1}\left[u^2E(A_3)\right]\\
&= E^{-1}\left[81u^{11}\right]=81\frac{x^9}{9!}=\frac{x^9}{4480}.
\end{aligned}$$

Thus, the approximate solution with the first five terms is obtained as follows.

$$\begin{aligned} y &\approx y_0 + y_1 + y_2 + y_3 + y_4 \\ &= x + \frac{x^3}{3} - \frac{7}{60}x^5 + \frac{61x^7}{2520} - \frac{547x^9}{181440} \\ &\quad -\frac{x^3}{2} + \frac{x^5}{20} + \frac{47}{840}x^7 - \frac{89x^9}{60480} \\ &\quad +\frac{3x^5}{40} - \frac{3x^7}{40} - \frac{523x^9}{20160} \\ &\quad -\frac{3x^7}{560} + \frac{29x^9}{960} \\ &\quad +\frac{x^9}{4480} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} = \phi_5(x). \end{aligned}$$

Above equation is first five terms of Maclaurin expansion of sinx which is a solution to Eq (4.1). by taking elzaki transform of the first five terms, we obtain

$$E\left[\phi_5(x)\right] = u^3 - u^5 + u^7 - u^9 + u^{11}.$$

All of the $[L/M]$ pade approximation of $E\left[\phi_5(x)\right]$, get $[L/M] = \frac{u^3}{1+u^2}$.
By applying inverse Elzaki transform to $[L/M]$, we obtain :
$E^{-1}\left[L/M\right] = E^{-1}\left[\frac{u^3}{1+u^2}\right] = \sin x.$
Thus a complete solution is obtained.

Example 4.2. $[5, 11]$ *Consider the Duffing's equation :* $\alpha = \beta = \gamma = 1.$

$$y'' + y' + y + y^3 = \cos^3 x - \sin x \tag{4.2}$$
$$y(0) = 1, y'(0) = 0.$$

The exact solution is $y(x) = \cos x$.

$$h(x) = \cos^3 x - \sin x \approx 1 - x - \frac{3x^2}{2} + \frac{x^3}{6} + \frac{7x^4}{8}.$$

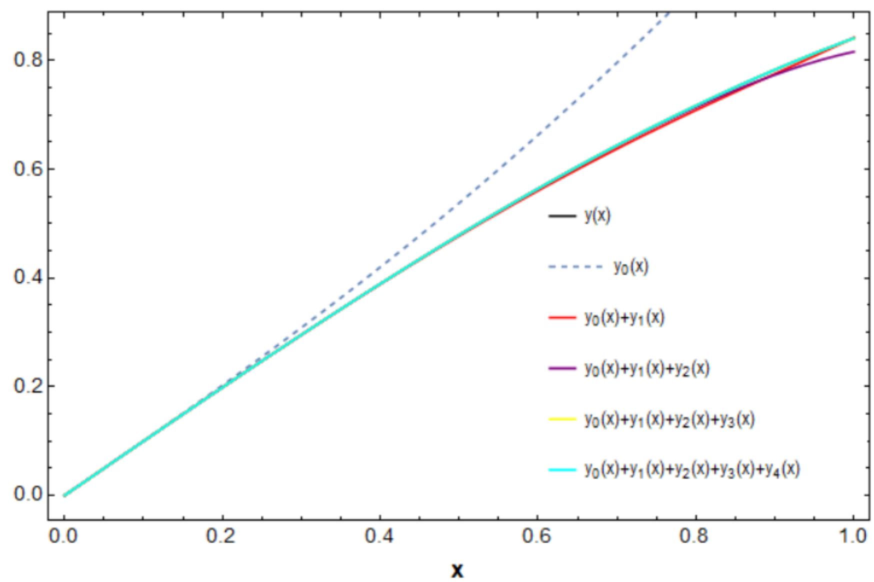

Figure 1: Solution of Equation (4.1) by EADM.

$$
\begin{aligned}
E\left[f(x)\right] &\approx E\left[1-x-\frac{3x^2}{2}+\frac{x^3}{6}+\frac{7x^4}{8}\right]\\
&\approx u^2-u^3-3u^4+u^5+21u^6
\end{aligned}
$$

$$
\begin{aligned}
y_0 &= E^{-1}\left[u^2y(0)+u^3y'(0)+\alpha u^3y(0)+u^2E\left(f(x)\right)\right]\\
&= E^{-1}\left[u^2+u^3+u^4-u^5-3u^6\right]\\
&= 1+x+\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^4}{8}
\end{aligned}
$$

$$A_0=(y_0)^3\approx\frac{7}{8}x^4+\frac{7}{2}x^3+\frac{9}{2}x^2+3x+1$$

$$E(A_0)=u^2+3u^3+9u^4+21u^5+21u^6$$

$$
\begin{aligned}
y_1 &= E^{-1}\left[-\left(u+u^2\right)E\left(y_0\right)-u^2E(A_0)\right] \\
&\approx E^{-1}\left[-u^3-3u^4-5u^5-9u^6\right] \\
&= -x-\frac{3x^2}{2}-\frac{5}{6}x^3-\frac{9}{24}x^4 \\
y_2 &= E^{-1}\left[-\left(u+u^2\right)E\left(y_1\right)-u^2E(A_1)\right] \\
&\approx E^{-1}\left[u^4+7u^5+29u^6\right] \\
&= \frac{x^2}{2}+\frac{7x^3}{6}+\frac{29x^4}{24} \\
y_3 &= E^{-1}\left[-\left(u+u^2\right)E\left(y_2\right)-u^2E(A_2)\right] \\
&= E^{-1}\left[-u^5-17u^6-147u^7-1009u^8\right] \\
&= -\frac{x^3}{6}-\frac{17x^4}{24} \\
y_4 &= E^{-1}\left[-\left(u+u^2\right)E\left(y_3\right)-u^2E(A_3)\right] \\
&\approx E^{-1}\left[u^6+45u^7+695u^8\right] \\
&= \frac{x^4}{24}.
\end{aligned}
$$

Thus, the approximate solution with the first five terms is obtained as follows.

$$
\begin{aligned}
y &\approx y_0+y_1+y_2+y_3+y_4 \\
&= 1+x+\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^4}{8} \\
&\quad -x-\frac{3x^2}{2}-\frac{5}{6}x^3-\frac{9}{24}x^4 \\
&\quad +\frac{x^2}{2}+\frac{7x^3}{6}+\frac{29x^4}{24} \\
&\quad -\frac{x^3}{6}-\frac{17x^4}{24} \\
&\quad +\frac{x^4}{24} \\
&= 1-\frac{x^2}{2!}+\frac{x^4}{4!}=\phi_5(x).
\end{aligned}
$$

Above equation is first five terms of Maclaurin expansion of $\cos x$ which is a solution to Eq (4.2). by taking elzaki transform of the first five terms, we obtain

$$E\left[\phi_5(x)\right] = u^2 - u^4 + u^6.$$

All of the $[L/M]$ pade approximation of $E\left[\phi_5(x)\right]$, get $[L/M] = \frac{u^2}{1+u^2}$.
By applying the inverse Elzaki transform to $[L/M]$, we obtain
$E^{-1}\left[L/M\right] = E^{-1}\left[\frac{u^2}{1+u^2}\right] = \cos x$
Thus a complete solution is obtained.

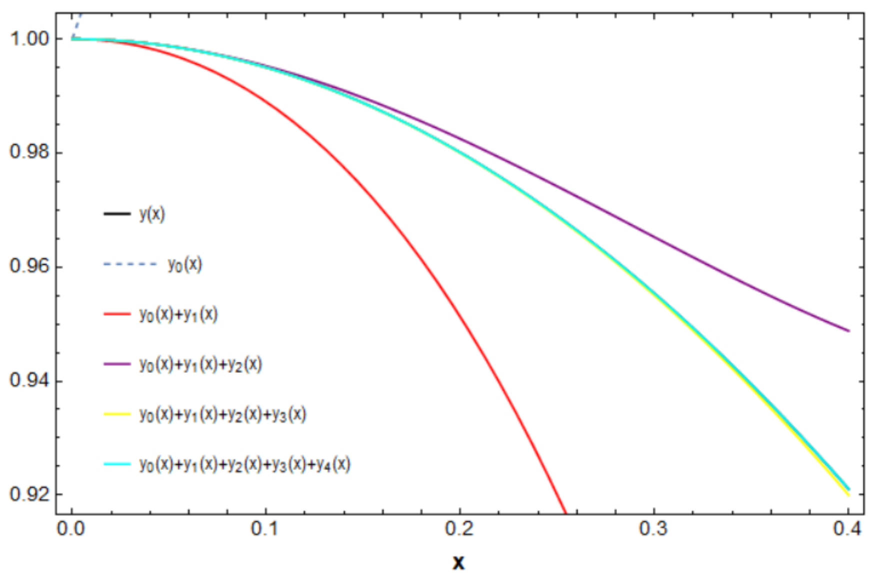

Figure 2: Solution of Equation (4.2) by EADM.

Example 4.3. $[5, 11]$*Consider the Duffing's equation :* $\alpha = 2, \beta = 1, \gamma = 8.$

$$y'' + 2y' + y + 8y^3 = e^{-3x} \tag{4.3}$$
$$y(0) = \frac{1}{2}, y'(0) = -\frac{1}{2}$$

The exact solution is $y(x) = \frac{e^{-x}}{2}$.

$$h(x) = e^{-3x} = 1 - 3x + \frac{9x^2}{2} - \frac{27x^3}{6}.$$

$$
\begin{aligned}
y_0 &= E^{-1}\left[u^2 y(0) + u^3 y'(0) + u^3 + u^2 E\left(1 - 3x + \frac{9x^2}{2} - \frac{27x^3}{6}\right)\right] \\
&= E^{-1}\left[\frac{u^2}{2} - \frac{u^3}{2} + u^3 + u^2\left(u^2 - 3u^3 + 9u^4 - 27u^5\right)\right] \\
&= E^{-1}\left[\frac{u^2}{2} + \frac{u^3}{2} + u^4 - 3u^5\right] = \frac{1}{2} + \frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{2}
\end{aligned}
$$

$$
\begin{aligned}
y_1 &= E^{-1}\left[-\left(2u + u^2\right) E\left(y_0\right) - 8u^2 E(A_0)\right] \\
&= E^{-1}\left[-\left(2u + u^2\right)\left(\frac{u^2}{2} + \frac{u^3}{2} + u^4 - 3u^5\right) - 8u^2 E\left(\frac{1}{8} + \frac{3}{8}x + \frac{3x^2}{4} + \frac{x^3}{2}\right)\right] \\
&= E^{-1}\left[-\left(2u + u^2\right)\left(\frac{u^2}{2} + \frac{u^3}{2} - u^4 - 3u^5\right) - 8u^2\left(-6u^5 + \frac{3u^3}{8} + \frac{u^2}{8}\right)\right] \\
&= E^{-1}\left[-u^3 - \frac{5u^4}{2} - \frac{11u^5}{2}\right] = -x - \frac{5x^2}{4} - \frac{11x^3}{12}
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= E^{-1}\left[-\left(2u + u^2\right) E\left(y_1\right) - 8u^2 E(A_1)\right] \\
&= E^{-1}\left[-\left(2u + u^2\right)\left(-u^3 - \frac{5u^4}{2}\right) - 8u^2 E\left(-\frac{77}{16}x^3 - \frac{39}{16}x^2 - \frac{3}{4}x\right)\right] \\
&= E^{-1}\left[-\left(2u + u^2\right)\left(-u^3 - \frac{5u^4}{2} - \frac{3u^5}{2}\right) - 8u^2\left(-\frac{231u^5}{8} - \frac{39u^4}{8} - \frac{3u^3}{4}\right)\right] \\
&= E^{-1}\left[2u^4 + 12u^5\right] = x^2 + 2x^3
\end{aligned}
$$

$$
\begin{aligned}
y_3 &= E^{-1}\left[-\left(2u + u^2\right)\left(2u^4 + 12u^5\right) - 8u^2 E(A_2)\right] \\
&= E^{-1}\left[-4u^5\right] = -\frac{2x^3}{3}
\end{aligned}
$$

Thus, the approximate solution with the first four terms is obtained as follows.

$$
\begin{aligned}
y &\approx y_0 + y_1 + y_2 + y_3 \\
&\approx \frac{1}{2} + \frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{2} - x - \frac{5x^2}{4} - \frac{11x^3}{12} + x^2 + 2x^3 - \frac{2x^3}{3} \\
&\approx \frac{1}{2} - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{12} = \phi_4(x).
\end{aligned}
$$

Above equation is first five terms of Maclaurin expansion of $\frac{e^{-x}}{2}$ which is a solution to Eq (4.3), by taking elzaki transform of the first four terms, we obtain :

$$E\left[\phi_4(x)\right] = \frac{u^2}{2} - \frac{u^3}{2} + \frac{u^4}{2} - \frac{u^5}{2}.$$

All of the $[L/M]$ pade approximation of $E\left[\phi_4(x)\right]$, get $[L/M] = \frac{u^2}{2(1+u)}$.
By applying the inverse Elzaki transform to $[L/M]$, we obtain :
$E^{-1}\left[L/M\right] = E^{-1}\left[\frac{u^2}{2(1+u)}\right] = \frac{e^{-x}}{2}$.
Thus a complete solution is obtained.

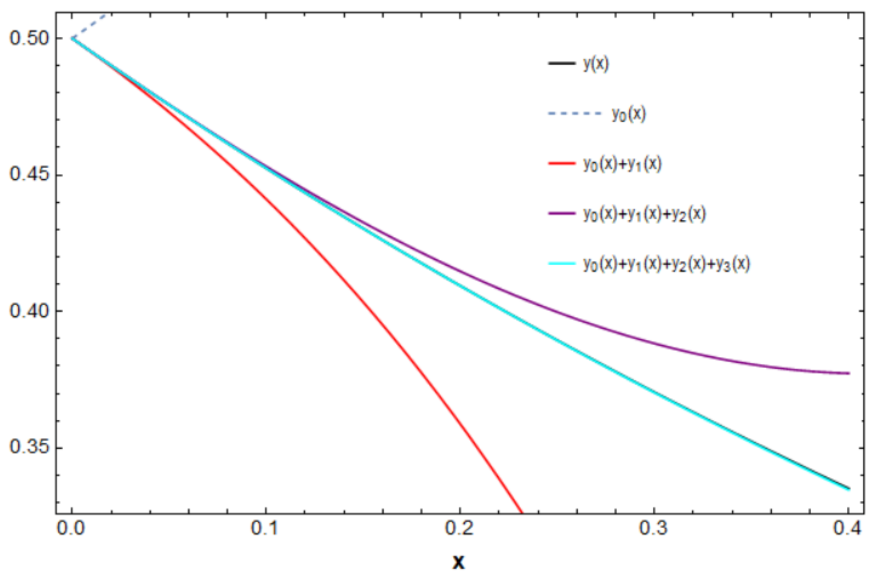

Figure 3: Solution of Equation (4.3) by EADM.

5 Conclusion

In this paper, a new technique (Elzaki Adomian Decomposition Method) was created to solve Duffing Equation. In addition, we observed that the results obtained by EADM were very consistent compared with other methods.

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