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Some unique common fixed point results in $b_v(s)$-metric spaces

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Abstract

In this paper we have established some unique common fixed point theorems on $b_v(s)$- metric spaces which are the extensions of theorems given in [11] and [4]. Some basic definitions, properties and lemmas are given in the introduction and preliminaries part. Some corollaries are also given on the basis of the result.

1 Introduction

After the famous Banach Contraction Principle (BCP), fixed point theorems have been developing and establishing in various metric spaces under different type of contractive conditions. In 1993, Czerwik [9] introduced the concept of b-metric space. In 2015, Jleli and Samet [14] gave a generalization on generalized metric spaces, which covers usual metric space, b-metric space and some other metric spaces also and then they established some fixed point results on that spaces. In

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2017, Mitrovic and Radenovic [16] gave more generalizations on rectangular b-metric spaces and introduced $b_v(s)$- metric space is as follows:

Definition 1.1. $\{$ *[11], [16]*$\}$. *Let X be a non-empty set, $s \geq 1$ be a real number, $v \in \mathbb{N}$ and $d : X \times X \to [0, \infty)$ be a function. Then d is said to be a $b_v(s)$- metric on X if for all $x, y, z \in X$ and for all distinct points $y_1, y_2, ..., y_v \in X$, each of them different from x and z such that the following conditions hold:*

1. $d(x, y) = 0$ *if and only if* $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq s[d(x, y_1) + d(y_1, y_2) + ... + d(y_v, z)]$.

The pair (X, d) is called $b_v(s)$-metric space.

Example 1.1. *[19]. Consider $X = \{\frac{1}{p} |\ p \in \mathbb{N}, p \geq 2\}$. Define $d : X \times X \to [0, \infty)$ by*

$$d(\frac{1}{p}, \frac{1}{q}) = \begin{cases} |p - q|, & if \ \ |p - q| \neq 1 \\ \frac{1}{2}, & if \ \ |p - q| = 1. \end{cases}$$

Then (X, d)- is a $b_3(3)$-metric space.

2 Preliminaries

Definition 2.1. *[16]. Let (X, d)-be a $b_v(s)$-metric space with $\{x_n\} \subset X$ be a sequence and $x \in X$. Then*

1. *The sequence $\{x_n\}$ is said to be convergent to x if for each $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n > N_0$.*
2. *The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, d) if for every $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $n, m > N(\epsilon)$.*
3. *(X, d) is said to be complete $b_v(s)$-metric space if for every Cauchy sequence in X converges to some $x \in X$.*

Definition 2.2. *[2]. Let f and g be two self maps on a set X (i.e, $f, g : X \to X$). If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g; and w is called a point of coincidence of f and g.*

Definition 2.3. *[2]. A pair of self maps $\{f, g\}$ defined on X is called weakly compatible if they commute at coincidence points.*

Definition 2.4. *[2]. Let f and g be mappings from a $b_v(s)$ metric space (X, d) into itself. Then f and g are said to be compatible mappings on X if $\lim_{m,n\to\infty} d(fgx_n, gfx_n)$ $= 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some point $t \in X$.*

Definition 2.5. *[4]. Let X be a non-empty set and $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. The mapping T is said to be α-admissible if $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1,\ \forall\, x, y \in X$.*

Definition 2.6. *[10]. Let X be a non-empty set and $T : X \to X$ and $\alpha : X\times X \to [0, \infty)$ be two mappings.The mapping T is said to be triangular α-admissible if*

1. $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1,\ \forall\, x, y \in X$.
2. $\alpha(x, z) \geq 1, \alpha(z, y) \geq 1 \Rightarrow \alpha(x, y) \geq 1,\ \forall\, x, y, z \in X$.

The following classes of functions are given in [11, 14].

Definition 2.7. $\{$ *[11], [4]*$\}$. *Γ_s be the class of functions defined by*
$\Gamma_s := \{\xi : [0, \infty) \to [0, \frac{1}{s})|$ *(i)* $\lim_{n\to\infty} \xi(t_n) = \frac{1}{s}$ *whenever* $\lim_{n\to\infty} t_n = 0$, *holds for some* $s \geq 1\}$.
If $s = 1$, we obtain the well-known class Γ of all Geragthy type contractive mappings given in [13].

Definition 2.8. $\{$ *[11], [4]*$\}$. *Ψ be the class of functions defined by*
$\Psi := \{\psi : [0, \infty) \to [0, \infty)|$ *(i) continuous, (ii) non-decreasing and (iii)* $\lim_{n\to\infty} \psi^n(t)$ $= 0$ *for all $t > 0$, where ψ^n denotes the n-th iteration of* $\psi\}$.

Note: If $\psi \in \Psi$, then (i) $\psi(t) < t$ for all $t > 0$, (ii) $\psi(0) = 0$.

In 2016, Roshan et al. [20] proved fixed point results in b-metric spaces by using Geraghty-weak contractions and later in 2021, Asim et al. [6] proved fixed point results for Geraghty-weak contractions in ordered partial rectangular b-metric spaces.

Inspiring and motivating the results of Roshan et al. [20], Asim et al. [6], Dosenovic et al. [11] and Arshad et al. [5] we have established the following theorems on $b_v(s)$- metric spaces which are the extensions of theorems given in [11] and [4].

3 Main results

Lemma 3.1. *Let (X, d)-be a $b_v(s)$-metric space and $d_n := d(x_n, x_{n-1})$. If the sequence $\{d_n\}$ is monotonic decreasing such that $d_n < \frac{1}{s}d_{n-1}$ with $s > 1$ and $\lim_{n\to\infty} d_n = 0$, then $\{x_n\}$ is a Cauchy sequence in X.*

Proof: We have to prove that $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.
Now for any $n, m \in \mathbb{N}$ with $m > n$ and $n_0 \in \mathbb{N}$, then we have

$$\begin{aligned}
&d(x_n, x_m)\\
&\leq s\Big[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+v-3}, x_{n+v-2}) + d(x_{n+v-2}, x_{n+n_0})\\
&+ d(x_{n+n_0}, x_{m+n_0}) + d(x_{m+n_0}, x_m)\Big]\\
&< s\Big[\Big(\frac{1}{s^n} + \frac{1}{s^{n+1}} + ... + \frac{1}{s^{n+v-3}}\Big)d(x_0, x_1) + \frac{1}{s^n}d(x_{v-2}, x_{n_0}) + \frac{1}{s^{n_0}}d(x_n, x_m)\\
&+ \frac{1}{s^m}d(x_{n_0}, x_0)\Big]
\end{aligned}$$

$$\begin{aligned}
\text{or, } &\Big[1 - \frac{1}{s^{n_0-1}}\Big]d(x_n, x_m) < \frac{1}{s^{n-1}}\left[\frac{1 - \frac{1}{s^{v-2}}}{1 - \frac{1}{s}}\right]d(x_0, x_1) + \frac{1}{s^{n-1}}d(x_{v-2}, x_{n_0})\\
&+ \frac{1}{s^{m-1}}d(x_0, x_{n_0})
\end{aligned} \tag{3.1}$$

Taking limit as $n, m \to \infty$ on both sides of (3.1) we have, $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

Therefore, $\{x_n\}$ is a Cauchy sequence in X.

Theorem 3.1. *Let (X, d) be a complete $b_v(s)$- metric space with a constant $s > 1$. Suppose that f, g, S and T are self maps on (X, d) with $f(X) \subset T(X), g(X) \subset S(X)$ such that for $\xi \in \Gamma_s, \psi \in \Psi$, the following conditions hold:*

$$sd(fx, gy) \leq \xi(M_1(x, y))\psi\{M(x, y)\} + LM_2(x, y), \quad \forall\, x, y \in X; \tag{3.2}$$

where,

$$M_1 = max\{d(fx, Sx), d(Sx, Ty), d(gy, Ty)\}; \tag{3.3}$$

$$M_2 = \mathit{min}\{d(fx, Sx), d(Sx, Ty), d(gy, Ty)\}; \tag{3.4}$$

$$M_3 = \mathit{max}\left\{d(Sx, Ty), \frac{d(fx, Ty)d(Sx, gy)}{a + d(Sx, Ty)}\right\}, \mathit{where}, \quad a \geq 1; \tag{3.5}$$

$$M = \alpha M_1 + \beta M_2 + \gamma M_3; \tag{3.6}$$

with, $\alpha + \beta + \gamma + sL < 1$, $\alpha \geq 0, \beta \geq 0, \gamma \geq 0, L \geq 0$;
and either

1. $\{f, S\}$ *is compatible, either* f *or* S *is continuous and* $\{g, T\}$ *is weakly compatible;*
   or,

2. $\{g, T\}$ *is compatible, either* g *or* T *is continuous and* $\{f, S\}$ *is weakly compatible;*
   or,

3. *The pairs* $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible and one of the ranges of* $f(X), g(X), S(X)$ *and* $T(X)$ *is closed.*

Then f, g, S *and* T *have a unique common fixed point in* X.

Proof: Let x_0 be an arbitrary point in X.We choose a point $x_1 \in X$ such that $y_0 = fx_0 = Tx_1$. This can be done, since the range of T contains the range of f. Similarly, a point $x_2 \in X$ can be chosen such that $y_1 = gx_1 = Sx_2$. Continuing these process, we can obtain a sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = gx_{2n+1} = Sx_{2n+2}.$$

First, we show that $\{y_n\}$ is a Cauchy sequence in X.
We consider two cases:
Case-I: Assume for some $n \in \mathbb{N}$, $y_n = y_{n+1}$ implies $y_{n+1} = y_{n+2}$. If not then, $d(y_{n+1}, y_{n+2}) > 0$.
Now for n is odd i.e., for $n = 2m - 1$, $m \in \mathbb{N}$, we have,

$$y_{2m-1} = y_{2m} \tag{3.7}$$

Now from (3.3),(3.4),(3.5) and (3.6) we have

$$\begin{aligned} &M_1(x_{2m}, x_{2m+1}) \\ &= \max\{d(fx_{2m}, Sx_{2m}), d(Sx_{2m}, Tx_{2m+1}), d(gx_{2m+1}, Tx_{2m+1})\} \\ &= \max\{d(y_{2m}, y_{2m-1}), d(y_{2m-1}, y_{2m}), d(y_{2m+1}, y_{2m})\} \\ &= d(y_{2m+1}, y_{2m}) \end{aligned} \tag{3.8}$$

$$\begin{aligned} &M_2(x_{2m}, x_{2m+1}) \\ &= \min\{d(fx_{2m}, Sx_{2m}), d(Sx_{2m}, Tx_{2m+1}), d(gx_{2m+1}, Tx_{2m+1})\} \\ &= \min\{d(y_{2m}, y_{2m-1}), d(y_{2m-1}, y_{2m}), d(y_{2m+1}, y_{2m})\} \\ &= 0 \end{aligned} \tag{3.9}$$

$$\begin{aligned} &M_3(x_{2m}, x_{2m+1}) \\ &= \max\left\{d(Sx_{2m}, Tx_{2m+1}), \frac{d(fx_{2m}, Tx_{2m+1})d(gx_{2m+1}, Sx_{2m})}{a + d(Tx_{2m+1}, Sx_{2m})}\right\} \\ &= \max\left\{d(y_{2m-1}, y_{2m}), \frac{d(y_{2m}, y_{2m})d(y_{2m+1}, y_{2m-1})}{a + d(y_{2m}, y_{2m-1})}\right\} \\ &= 0 \end{aligned} \tag{3.10}$$

Therefore, $$M(x_{2m}, x_{2m+1}) = \alpha d(y_{2m+1}, y_{2m}) \tag{3.11}$$

Now from (3.2) we have

$$sd(y_{2m}, y_{2m+1}) \leq \xi\{M_1(x_{2m}, x_{2m+1})\}\psi\{M(x_{2m}, x_{2m+1})\}+ LM_2(x_{2m}, x_{2m+1}) \leq \frac{1}{s}\alpha d(y_{2m+1}, y_{2m})$$

or, $$\left(\frac{s^2-\alpha}{s}\right)d(y_{2m+1}, y_{2m}) \leq 0, \quad \text{which is a contradiction.} \tag{3.12}$$

Hence, $$y_{2m-1} = y_{2m} \text{ implies } y_{2m} = y_{2m+1} \tag{3.13}$$

Similarly, for n is even i.e., for $n = 2m,\ m \in \mathbb{N}$, we can prove that

$$y_{2m} = y_{2m+1} \text{ implies } y_{2m+1} = y_{2m+2} \tag{3.14}$$

Therefore, from (3.13) and (3.14) we have

$$y_n = y_{n+1} \text{ implies } y_{n+1} = y_{n+2}, \quad \forall n \in \mathbb{N}. \tag{3.15}$$

Continuing in this manner we can show that $y_n = y_{n+1}$ implies $y_n = y_{n+k}, \ \forall\, k = 1, 2, 3, ...$
Therefore $\{y_n\}$ becomes a constant sequence and a Cauchy one.
Case-II: Assume $y_n \neq y_{n+1}, \quad \forall\, n \in \mathbb{N}$.
For n is odd. Let $n = 2m-1, \quad m \in \mathbb{Z}^+$, From (3.3), (3.4), (3.5) and (3.6) we have

$$\begin{aligned} &M_1(x_{2m}, x_{2m+1}) \\ &= \max\{d(fx_{2m}, Sx_{2m}), d(Sx_{2m}, Tx_{2m+1}), d(gx_{2m+1}, Tx_{2m+1})\} \\ &= \max\{d(y_{2m}, y_{2m-1}), d(y_{2m-1}, y_{2m}), d(y_{2m+1}, y_{2m})\} \end{aligned} \tag{3.16}$$

$$\begin{aligned} &M_2(x_{2m}, x_{2m+1}) \\ &= \min\{d(fx_{2m}, Sx_{2m}), d(Sx_{2m}, Tx_{2m+1}), d(gx_{2m+1}, Tx_{2m+1})\} \\ &= \min\{d(y_{2m}, y_{2m-1}), d(y_{2m-1}, y_{2m}), d(y_{2m+1}, y_{2m})\} \end{aligned} \tag{3.17}$$

$$\begin{aligned} &M_3(x_{2m}, x_{2m+1}) \\ &= \max\left\{d(Sx_{2m}, Tx_{2m+1}), \frac{d(fx_{2m}, Tx_{2m+1})d(gx_{2m+1}, Sx_{2m})}{a + d(Tx_{2m+1}, Sx_{2m})}\right\} \\ &= \max\left\{d(y_{2m-1}, y_{2m}), \frac{d(y_{2m}, y_{2m})d(y_{2m+1}, y_{2m-1})}{a + d(y_{2m}, y_{2m-1})}\right\} \end{aligned} \tag{3.18}$$

If $d(y_{2m+1}, y_{2m}) \geq d(y_{2m}, y_{2m-1})$, then

$$M(x_{2m}, x_{2m+1}) \leq \alpha d(y_{2m+1}, y_{2m}) + \beta d(y_{2m}, y_{2m-1}) + \gamma d(y_{2m-1}, y_{2m}).$$

Therefore, from (3.2) we have

$$\begin{aligned} sd(y_{2m}, y_{2m+1}) &< \frac{1}{s}[\alpha d(y_{2m+1}, y_{2m}) + (\beta+\gamma)d(y_{2m}, y_{2m-1})] + Ld(y_{2m}, y_{2m-1}) \\ &\text{or,} \quad (s - \frac{\alpha}{s})d(y_{2m}, y_{2m+1}) < (\frac{\beta+\gamma+sL}{s})d(y_{2m}, y_{2m-1}) \\ &\text{or,} \quad d(y_{2m}, y_{2m+1}) < (\frac{\beta+\gamma+sL}{s^2-\alpha})d(y_{2m}, y_{2m-1}), \\ &\text{which is a contradiction.} \\ &\qquad\qquad [\text{As} \quad \frac{\beta+\gamma+sL}{s^2-\alpha} < 1]. \end{aligned}$$

Hence, $$d(y_{2m+1}, y_{2m}) < d(y_{2m}, y_{2m-1}).$$

Now from (3.6) we have

$$M(x_{2m}, x_{2m+1}) = (\alpha+\gamma)d(y_{2m}, y_{2m-1}) + \beta d(y_{2m+1}, y_{2m})$$

Therefore, from (3.2) we have

$$sd(y_{2m}, y_{2m+1}) < \frac{1}{s}(\alpha+\gamma)d(y_{2m}, y_{2m-1}) + \frac{\beta}{s}d(y_{2m+1}, y_{2m}) + Ld(y_{2m+1}, y_{2m})$$

or, $d(y_{2m+1}, y_{2m}) < (\dfrac{\alpha+\gamma}{s^2-\beta-sL})d(y_{2m}, y_{2m-1})$.

$$\text{Hence,} \quad d(y_{2m+1}, y_{2m}) < \frac{1}{s}d(y_{2m}, y_{2m-1}), \quad \text{as } (\frac{\alpha+\gamma}{s^2-\beta-sL}) < \frac{1}{s}. \tag{3.19}$$

For n is even i.e., $n = 2m, \quad m \in \mathbb{Z}^+$,

$$\begin{aligned} &M_1(x_{2m+2}, x_{2m+1}) \\ &= \max\{d(fx_{2m+2}, Sx_{2m+2}), d(Sx_{2m+2}, Tx_{2m+1}), \\ &\quad d(gx_{2m+1}, Tx_{2m+1})\} \\ &= \max\{d(y_{2m+2}, y_{2m+1}), d(y_{2m+1}, y_{2m}), d(y_{2m+1}, y_{2m})\} \end{aligned} \tag{3.20}$$

$$\begin{aligned} &M_2(x_{2m+2}, x_{2m+1}) \\ &= \min\{d(fx_{2m+2}, Sx_{2m+2}), d(Sx_{2m+2}, Tx_{2m+1}), \\ &\quad d(gx_{2m+1}, Tx_{2m+1})\} \\ &= \min\{d(y_{2m+2}, y_{2m+1}), d(y_{2m+1}, y_{2m}), d(y_{2m+1}, y_{2m})\} \end{aligned} \tag{3.21}$$

$$\begin{aligned} &M_3(x_{2m+2}, x_{2m+1}) \\ &= \max\left\{d(Sx_{2m+2}, Tx_{2m+1}), \frac{d(fx_{2m+2}, Tx_{2m+1})d(gx_{2m+1}, Sx_{2m+2})}{a + d(Tx_{2m+1}, Sx_{2m+2})}\right\} \\ &= \max\left\{d(y_{2m+1}, y_{2m}), \frac{d(y_{2m+2}, y_{2m})d(y_{2m+1}, y_{2m+1})}{a + d(y_{2m+1}, y_{2m})}\right\} \end{aligned} \tag{3.22}$$

If $d(y_{2m+2}, y_{2m+1}) \geq d(y_{2m+1}, y_{2m})$, then from (3.6) we have
$M(x_{2m+2}, x_{2m+1}) \leq \alpha d(y_{2m+2}, y_{2m+1}) + \beta d(y_{2m+1}, y_{2m}) + \gamma d(y_{2m+1}, y_{2m})$

Therefore, from (3.2) we have

$$sd(y_{2m+2}, y_{2m+1}) < \frac{1}{s}[\alpha d(y_{2m+2}, y_{2m+1}) + \beta d(y_{2m+1}, y_{2m}) + \gamma d(y_{2m+1}, y_{2m})]$$
$$+ Ld(y_{2m+1}, y_{2m})$$
$$\text{or, } (s - \frac{\alpha}{s})d(y_{2m+2}, y_{2m+1}) < (\frac{\beta+\gamma}{s})d(y_{2m+1}, y_{2m}) + Ld(y_{2m+1}, y_{2m})$$
$$\text{or, } d(y_{2m+2}, y_{2m+1}) < \left(\frac{\beta+\gamma+sL}{s^2-\alpha}\right)d(y_{2m+1}, y_{2m}), \text{ which is a contradiction.}$$
$$[\text{as } (\frac{\beta+\gamma+sL}{s^2-\alpha}) < 1].$$

Hence, $d(y_{2m+2}, y_{2m+1}) < d(y_{2m+1}, y_{2m})$.
Now from (3.2) we have

$$sd(y_{2m+2}, y_{2m+1}) < \frac{1}{s}[\alpha d(y_{2m+1}, y_{2m}) + \beta d(y_{2m+2}, y_{2m+1}) + \gamma d(y_{2m+1}, y_{2m})]$$
$$+ Ld(y_{2m+2}, y_{2m+1})\text{or, } (s - \frac{\beta}{s} - L)d(y_{2m+2}, y_{2m+1}) < (\frac{\alpha+\gamma}{s})d(y_{2m+1}, y_{2m}))$$
$$\text{or, } d(y_{2m+2}, y_{2m+1}) < (\frac{\alpha+\gamma}{s^2-\beta-sL})d(y_{2m+1}, y_{2m})$$

$$d(y_{2m+2}, y_{2m+1}) < \frac{1}{s}d(y_{2m+1}, y_{2m}), \quad \text{as } (\frac{\alpha+\gamma}{s^2-\beta-sL}) < \frac{1}{s}. \tag{3.23}$$

From (3.19) and (3.23) we have

$$d(y_{n+1}, y_n) < \frac{1}{s}d(y_n, y_{n-1}), \qquad \forall \ n \in \mathbb{N}. \tag{3.24}$$

Claim: $\lim_{n\to\infty} d(y_n, y_{n-1}) = 0$.
As $\{d(y_n, y_{n-1})\}$ is non-negative and monotone decreasing so there exists $l \geq 0$ such that

$$\lim_{n\to\infty} d(y_n, y_{n-1}) = l.$$

Now using (3.2) we have

$$\begin{aligned} l &= \lim_{n\to\infty} d(y_{2n+1}, y_{2n+2}) \\ &\leq \lim_{n\to\infty} sd(y_{2n+1}, y_{2n+2}) \\ &\leq \lim_{n\to\infty} \{\xi\{M(y_{2n}, y_{2n+1})\}\psi\{M(y_{2n}, y_{2n+1})\} + \lim_{n\to\infty} LM(y_{2n}, y_{2n+1}) \\ &\leq \frac{(\alpha+\beta+\gamma+sL)l}{s} < l, \quad \text{which is a contradiction.} \end{aligned}$$

Hence, $l = 0$.

$$\text{Therefore,} \qquad \lim_{n\to\infty} d(y_n, y_{n-1}) = 0. \tag{3.25}$$

Since the sequence $\{d(y_n, y_{n-1})\}$ is monotonic decreasing such that $d_n < \frac{1}{s}d_{n-1}$ with $s > 1$ and $\lim_{n\to\infty} d(y_n, y_{n-1}) = 0$, then by Lemma 3.1 we conclude that $\{y_n\}$ is Cauchy in X.

Since (X, d) is complete, so there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

Condition-A: Assume S is continuous.

Since $\{f, S\}$ is compatible, we have $\lim_{n\to\infty} fSx_{2n+2} = \lim_{n\to\infty} Sfx_{2n+2} = Sz$.

Claim: $Sz = z$. If not then, $d(Sz, z) > 0$.

Now from (3.3),(3.4) and (3.5) we have

$$\begin{aligned} &M_1(Sx_{2n+2}, x_{2n+1}) \\ &= \max\{d(fSx_{2n+2}, SSx_{2n+1}), d(SSx_{2n+2}, Tx_{2n+1}), d(gx_{2n+1}, Tx_{2n+1})\}, \\ &M_2(Sx_{2n+2}, x_{2n+1}) \\ &= \min\{d(fSx_{2n+2}, SSx_{2n+1}), d(SSx_{2n+2}, Tx_{2n+1}), d(gx_{2n+1}, Tx_{2n+1})\}, \\ &M_3(Sx_{2n+2}, x_{2n+1}) \\ &= \max\left\{d(SSx_{2n+2}, Tx_{2n+1}), \frac{d(fSx_{2n+2}, Tx_{2n+1})d(gx_{2n+1}, SSx_{2n+2})}{a + d(Tx_{2n+1}, SSx_{2n+2})}\right\} \end{aligned}$$

Therefore,

$$\begin{aligned} &\lim_{n\to\infty} M(Sx_{2n+2}, x_{2n+1}) \\ &= \lim_{n\to\infty} \{\alpha M_1(Sx_{2n+2}, x_{2n+1}) + \beta M_2(Sx_{2n+2}, x_{2n+1}) + \gamma M_3(Sx_{2n+2}, x_{2n+1})\} \end{aligned}$$

$$\text{or,} \qquad \lim_{n\to\infty} M(Sx_{2n+2}, x_{2n+1}) = (\alpha+\gamma)d(Sz, z). \tag{3.26}$$

Now from (3.2) we have

$$sd(fSx_{2n+2}, gx_{2n+1}) \leq \xi\{M_1(Sx_{2n+2}, x_{2n+1})\}\psi\{M(Sx_{2n+2}, x_{2n+1})\}$$
$$+ LM_2(Sx_{2n+2}, x_{2n+1}).$$

$$\text{or,} \quad sd(fSx_{2n+2}, gx_{2n+1}) < \frac{1}{s}M(Sx_{2n+2}, x_{2n+1}) + LM_2(Sx_{2n+2}, x_{2n+1}). \tag{3.27}$$

Taking limit as $n \to \infty$ on both sides of (3.27) we have,

$$\begin{aligned} \lim_{n\to\infty} sd(fSx_{2n+2}, gx_{2n+1}) &\leq \lim_{n\to\infty} \frac{1}{s}M(Sx_{2n+2}, x_{2n+1}) \\ &\quad + \lim_{n\to\infty} LM_2(Sx_{2n+2}, x_{2n+1}) \\ \text{or,} \quad sd(Sz, z) &\leq (\frac{\alpha+\gamma}{s})d(Sz, z) + Ld(Sz, z) \\ \text{or,} \quad (s^2 - sL - \alpha - \gamma)d(Sz, z) &\leq 0, \quad \text{which is a contradiction as} \\ & (s^2 - sL - \alpha - \gamma) > 0. \end{aligned}$$

Hence, $Sz = z$.
Claim: $fz = z$. If not then, $d(fz, z) > 0$.
Now

$$\begin{aligned} M_1(z, x_{2n+1}) &= \max\{d(fz, Sz), d(Sz, Tx_{2n+1}), d(gx_{2n+1}, Tx_{2n+1})\} \\ M_2(z, x_{2n+1}) &= \min\{d(fz, Sz), d(Sz, Tx_{2n+1}), d(gx_{2n+1}, Tx_{2n+1})\} \\ M_3(z, x_{2n+1}) &= \max\left\{d(Sz, Tx_{2n+1}), \frac{d(fz, Tx_{2n+1})d(gx_{2n+1}, Sx_{2n+1})}{a + d(Tx_{2n+1}, Sz)}\right\}. \end{aligned}$$

Also from (3.2) we have

$$M(z, x_{2n+1}) = \alpha M_1(z, x_{2n+1}) + \beta M_2(z, x_{2n+1}) + \gamma M_3(z, x_{2n+1}). \tag{3.28}$$

Taking limit as $n \to \infty$ on both sides of (3.28), we have

$$\lim_{n\to\infty} M(z, x_{2n+1}) = \alpha d(fz, z). \tag{3.29}$$

Now from (3.2) we have $sd(fz, gx_{2n+1}) \leq \xi\{M_1(z, x_{2n+1})\}\psi\{M(z, x_{2n+1})\} +$

$LM_2(z, x_{2n+1})$

$$\text{or,} \quad sd(fz, gx_{2n+1}) < \frac{1}{s}M(z, x_{2n+1}) + LM_2(z, x_{2n+1}) \tag{3.30}$$

Taking limit as $n \to \infty$ on both sides of (3.30) we have

$$\lim_{n\to\infty} sd(fz, gx_{2n+1}) \leq \lim_{n\to\infty} \frac{1}{s}M(z, x_{2n+1}) + \lim_{n\to\infty} LM_2(z, x_{2n+1}).$$

or, $sd(fz, z) \leq \frac{1}{s}\alpha d(fz, z)$, which is a contradiction.
Hence, $fz = z$.
Therefore, $fz = Sz = z$.
Now since $f(X) \subset T(X)$, there exists a point $w \in X$ such that $fz = Tw$.
Claim: $gw = Tw$. If not then, $d(gw, Tw) > 0$.

$$\begin{aligned} M_1(z, w) &= \max\{d(fz, Sz), d(Sz, Tw), d(gw, Tw)\} = d(gw, Tw); \\ M_2(z, w) &= \min\{d(fz, Sz), d(Sz, Tw), d(gw, Tw)\} = 0; \\ M_3(z, w) &= \max\{d(Sz, Tw), \frac{d(fz, Tw)d(gw, Sz)}{a + d(Tw, Sz)}\} = 0; \\ \text{and } M(z, w) &= \alpha M_1(z, w) + \beta M_2(z, w) + \gamma M_3(z, w) \\ &= \alpha d(gw, Tw). \end{aligned}$$

Now from (3.2), we have

$$\begin{aligned} &sd(Tw, gw) = sd(fz, gw) \leq \xi\{M_1(z, w)\}\psi\{M(z, w)\} + LM_2(z, w) \\ &\text{or,} \quad sd(Tw, gw) < \frac{1}{s}M(z, w) + LM_2(z, w)\} \\ &\text{or,} \quad sd(Tw, gw) < \frac{1}{s}\alpha d(gw, Tw), \quad \text{which is a contradiction.} \end{aligned}$$

Hence, $d(Tw, gw) = 0$.
Since $\{g, T\}$ is weakly compatible so, $gz = gfz = gTw = Tgw = Tfz = Tz$.
Hence z is a coincidence point of g and T.
Claim: $gz = z$. If not then, $d(gz, z) > 0$.

Now

$$M_1(z,z)=0,\quad M_2(z,z)=0,\quad M_3(z,z)=0,\ \text{ and } M(z,z)=0.$$

Then, $$sd(fz,gz)\le \xi\{M_1(z,z)\}\psi\{M(z,z)\}+LM_2(z,z)$$
$$\le 0,\quad \text{which is a contradiction.}$$

Hence, $gz=z$.
Therefore, $fz=gz=Sz=Tz=z$.
Condition-B: The proof is same as in **condition-A**.
Condition-C: Assuming T is closed. Then there exists $u\in X$ such that $z=Tu$.
Claim: $gu=z$. If not then, $d(gu,z)>0$.
Now

$$M_1(x_{2n},u)=\max\{d(fx_{2n},Sx_{2n}),d(Sx_{2n},Tu),d(gu,Tu)\};$$
$$M_2(x_{2n},u)=\min\{d(fx_{2n},Sx_{2n}),d(Sx_{2n},Tu),d(gu,Tu)\};$$
$$M_3(x_{2n},u)=\max\left\{d(Sx_{2n},Tu),\frac{d(fx_{2n},Tu),d(gu,Sx_{2n})}{a+d(Tu,Sx_{2n})}\right\}.$$

$$M(x_{2n},u)=\alpha M_1(x_{2n},u)+\beta M_2(x_{2n},u)+\gamma M_3(x_{2n},u) \tag{3.31}$$

Taking limit as $n\to\infty$ on both sides of (3.31) we have

$$\lim_{n\to\infty} M(x_{2n},u)=\alpha d(gu,z) \tag{3.32}$$

Now from (3.2) we have

$$sd(fx_{2n},gu)\le \xi\{M_1(x_{2n},u)\}\psi\{M(x_{2n},u)\}+LM_2(x_{2n},u)$$

$$\text{or,}\quad sd(fx_{2n},gu)<\frac{1}{s}M(x_{2n},u)+LM_2(x_{2n},u) \tag{3.33}$$

Taking limit as $n\to\infty$ on both sides of (3.33) we have

$$\begin{aligned}&\lim_{n\to\infty} sd(fx_{2n},gu)\le \lim_{n\to\infty}\frac{1}{s}M(x_{2n},u)+\lim_{n\to\infty}LM_2(x_{2n},u)\\ &\text{or,}\ \ sd(z,gu)\le \frac{1}{s}\alpha d(gu,z)\\ &\text{or,}\ \ (s^2-\alpha)d(z,gu)\le 0,\qquad \text{which is a contradiction.}\end{aligned} \tag{3.34}$$

Hence, $gu=z$.

Therefore, $gu = Tu = z$.
Since $\{g, T\}$ is weakly compatible, then we have $gz = gTu = Tgu = Tz$
Claim: $gz = z$. If not then, $d(gz, z) > 0$.

$$\begin{aligned} M_1(x_{2n}, z) &= \max\{d(fx_{2n}, Sx_{2n}), d(Sx_{2n}, Tz), d(gz, Tz)\}; \\ M_2(x_{2n}, z) &= \min\{d(fx_{2n}, Sx_{2n}), d(Sx_{2n}, Tz), d(gz, Tz)\}; \\ M_3(x_{2n}, z) &= \max\left\{d(Sx_{2n}, Tz), \frac{d(fx_{2n}, Tz), d(gz, Sx_{2n})}{a + d(Tz, Sx_{2n})}\right\}. \end{aligned} \tag{3.35}$$

There fore, $$M(x_{2n}, z) = \alpha M_1(x_{2n}, z) + \beta M_2(x_{2n}, z) + \gamma M_3(x_{2n}, z). \tag{3.36}$$

Taking limit as $n \to \infty$ on both sides of (3.36) we have

$$\lim_{n\to\infty} M(x_{2n}, z) = \alpha d(z, Tz) + \gamma d(z, Tz). \tag{3.37}$$

Now from (3.2) we have
$sd(fx_{2n}, gz) \leq \xi\{M_1(x_{2n}, z)\}\psi\{M(x_{2n}, z)\} + LM_2(x_{2n}, z)$

$$\text{or,} \quad sd(fx_{2n}, gz) < \frac{1}{s}M(x_{2n}, z) + LM_2(x_{2n}, z). \tag{3.38}$$

Taking limit as $n \to \infty$ on both sides of (3.38) we have

$$\begin{aligned} &\lim_{n\to\infty} sd(fx_{2n}, gz) \leq \lim_{n\to\infty} \frac{1}{s}M(x_{2n}, z) + \lim_{n\to\infty} LM_2(x_{2n}, z) \\ &\text{or,} \quad sd(z, gz) \leq \frac{1}{s}(\alpha + \gamma)d(z, gz) \qquad [\text{As } gz = Tz] \\ &\text{or,} \quad \left(\frac{s^2 - \alpha - \gamma}{s}\right)d(z, gz) \leq 0, \quad \text{which is a contradiction.} \end{aligned}$$

Hence, $gz = z$.
Therefore, $gz = Tz = z$.
Since $g(X) \subset S(X)$, there exists $v \in X$ such that $gz = Sv$. Then $gz = Sv = Tz = z$.
Claim: $fv = Sv$. If not then, $d(fv, Sv) > 0$.

Now

$$\begin{aligned}
M_1(v,z) &= \max\{d(fv,Sv), d(z,Tz), d(gz,Tz)\}\\
M_2(v,z) &= \min\{d(fv,Sv), d(Sv,Tz), d(gz,Tz)\}\\
M_3(v,z) &= \max\{d(Sv,Tz), \frac{(d(fv,Tz)d(gz,Sv))}{(a+d(Tz,Sv))}\}\\
\text{and } M(v,z) &= \alpha M_1(v,z) + M_2(v,z) + \gamma M_3(v,z)\\
&= \alpha d(fv,z).
\end{aligned}$$

Therefore from (3.2), we have

$$\begin{aligned}
& sd(fv,Sv) = sd(fv,gz) \leq \xi\{M_1(v,z)\}\psi\{M(v,z)\} + LM_2(v,z)\\
& \text{or, } sd(fv,z) \leq \frac{1}{s}M(v,z)\\
& \text{or, } sd(fv,z) \leq \frac{1}{s}\alpha d(fv,z)\\
& \text{or, } (\frac{s^2-\alpha}{s})d(fv,z) \leq 0, \text{ which is a contradiction.}
\end{aligned}$$

Hence, $fv = z = Sv.$

Since $\{f,S\}$ is weakly compatible, so $fz = fTz = fSv = Sfv = Sz.$

Claim: $fz = z.$

$$\begin{aligned}
M_1(z,z) &= \max\{d(fz,Sz), d(Sz,Tz), d(gz,Tz)\} = d(fz,z)\\
M_2(z,z) &= \min\{d(fz,Sz), d(Sz,Tz), d(gz,Tz)\} = 0\\
M_3(z,z) &= \max\left\{d(Sz,Tz), \frac{d(fz,Tz)d(gz,Sz)}{a+d(Tz,Sz)}\right\} = 0
\end{aligned}$$

Therefore, $M = \alpha M(z,z) = \alpha d(fz,z).$

Now

$$\begin{aligned}
& sd(fz,gz) = sd(fz,gz) \leq \xi\{M_1(z,z)\}\psi\{M(z,z)\} + LM_2(z,z) \leq \frac{1}{s}M(z,z)\\
& \text{or, } sd(fz,z) \leq \frac{1}{s}\alpha d(fz,z)\\
& \text{or, } (\frac{s^2-\alpha}{s})d(fz,z) \leq 0, \quad \text{which is a contradiction.}
\end{aligned}$$

Hence, $fz = z.$

Therefore, $fz = gz = Sz = Tz = z.$
Uniqueness: Now we show that z is the unique common fixed point of f, g, s and T. If possible let there exists an another $u \in X$ such that $fu = gu = Su = Tu = u$ with $u \neq z$.
Now

$$\begin{aligned} M_1(u,z) &= \max\{d(fu,Su), d(Su,Tz), d(gz,Tz)\} = d(u,z); \\ M_2(u,z) &= \min\{d(fu,Su), d(Su,Tz), d(gz,Tz)\} = 0; \\ M_3(u,z) &= \max\left\{d(Su,Tz), \frac{d(fu,Tz)d(gz,Su)}{a+d(Tz,Su)}\right\} = d(u,z); \\ &\text{and } M(u,z) = (\alpha+\gamma)d(u,z). \end{aligned}$$

Now from (3.2) we have

$$\begin{aligned} & sd(u,z) = sd(fu,gz) \leq \xi\{M_1(u,z)\}\psi\{M(u,z)\} + LM_2(u,z) \\ & \text{or, } \ sd(u,z) \leq \frac{1}{s}M(u,z) \\ & \text{or, } \ sd(u,z) \leq \frac{1}{s}(\alpha+\gamma)d(u,z) \\ & \text{or, } \ \left(\frac{s^2-\alpha-\gamma}{s}\right)d(u,z) \leq 0, \ \text{ which is a contradiction.} \end{aligned}$$

Hence, $u = z.$
Therefore, z is a unique common fixed point of f, g, S and T.

Corollary 3.1. *Let (X,d) be a complete $b_v(s)$- metric space with a constant $s > 1$. Suppose that f, g, S and T are self maps on (X,d) with $f(X) \subset T(X), g(X) \subset S(X)$ such that for $\xi \in \Gamma_s, \psi \in \Psi$, the following hold:*

$$sd(fx,gy) \leq \xi(M_1(x,y))\psi\{M(x,y)\}, \quad \forall \ x,y \in X; \tag{3.39}$$

where,

$$M_1 = \max\{d(fx,Sx), d(Sx,Ty), d(gy,Ty)\}; \tag{3.40}$$

$$M_2 = \min\{d(fx,Sx), d(Sx,Ty), d(gy,Ty)\}; \tag{3.41}$$

$$M_3 = \max\left\{d(Sx,Ty), \frac{d(fx,Ty)d(Sx,gy)}{a+d(Sx,Ty)}\right\}, \text{where} \quad a \geq 1; \tag{3.42}$$

$$M = \alpha M_1 + \beta M_2 + \gamma M_3; \tag{3.43}$$

with $\alpha+\beta+\gamma<1$, $\alpha\geq 0,\beta\geq 0,\gamma\geq 0$*;*
Also either

1. $\{f,S\}$ *is compatible, either* f *or* S *is continuous and* $\{g,T\}$ *is weakly compatible;*
   or,

2. $\{g,T\}$ *is compatible, either* g *or* T *is continuous and* $\{f,S\}$ *is weakly compatible;*
   or,

3. *The pairs* $\{f,S\}$ *and* $\{g,T\}$ *are weakly compatible and one of the ranges of* $f(X),g(X),S(X)$ *and* $T(X)$ *is closed.*

Then f,g,S *and* T *have a unique common fixed point in* X.

Proof: Taking $L=0$ and proceeding same as in Theorem 1.1 we can prove the result.

Theorem 3.2. (X,d) *be a complete* $b_v(s)$*- metric space with constant* $s>1$ *and* $f,g:X\to X$ *be two mappings on* X *such that for* $\xi\in\Gamma_s,\psi\in\Psi$*, the following hold:*

$$sd(fx,gy)\leq\xi\{M(x,y)\}\psi\{M(x,y)\}+LM_1(x,y),\quad\forall\ x,y\ \in X;\tag{3.44}$$

where,
$$M(x,y)=\alpha\frac{d(x,fx)+d(y,gy)}{a+d(x,y)}+\beta max\{d(x,y),d(y,gy),d(x,fx)\}+\gamma min\{d(x,gy),d(y,fx)\},\quad \textit{where, } a\geq 1;\tag{3.45}$$

$$M_1(x,y)=min\{d(x,y),d(y,gy),d(x,fx)\}.\tag{3.46}$$

with $0\leq\alpha+\beta+\gamma\leq 1;\alpha\geq 0,\beta\geq 0,\gamma\geq 0$. *Then* f,g *have a unique common fixed point in* X.

Proof: Let x_0 be an arbitrary point in X. We define a sequence $\{x_n\}\subset X$ by

$$x_{2n+1}=fx_{2n} \text{ and } x_{2n+2}=gx_{2n+1}.$$

We consider two cases:
Case-I: Assume for some $n \in \mathbb{N}$, $x_n = x_{n+1}$ implies $x_{n+1} = x_{n+2}$.
If $x_n = x_{n+1}$, then for n is even i.e., for $n = 2m, m \in \mathbb{N}$ we have

$$x_{2m} = x_{2m+1}. \tag{3.47}$$

If $x_{2m+1} \neq x_{2m+1}$, then $d(x_{2m+1}, x_{2m+2}) > 0$,
Now

$$\begin{aligned} &M(x_{2m}, x_{2m+1}) \\ &= \alpha \frac{d(x_{2m}, fx_{2m}) + d(x_{2m+1}, gx_{2m+1})}{a + d(x_{2m}, x_{2m+1})} + \beta \text{max}\{d(x_{2m}, fx_{2m+1}), d(x_{2m+1}, gx_{2m+1}), \\ &\quad d(x_{2m}, fx_{2m})\} + \gamma \min\{d(x_{2m}, x_{2m+2}), d(x_{2m+1}, x_{2m+1})\} \\ &\leq (\alpha + \beta) d(x_{2m+1}, x_{2m+2}) \end{aligned} \tag{3.48}$$

Therefore, from (3.44) and using (3.48) we have

$$\begin{aligned} d(x_{2m+1}, x_{2m+2}) &\leq sd(fx_{2m}, gx_{2m+1}) \\ &\leq \xi\{M(x_{2m}, x_{2m+1})\}\psi\{M(x_{2m}, x_{2m+1})\} + LM_1(x_{2m}, x_{2m+1}) \\ &< \frac{1}{s} M(x_{2m}, x_{2m+1}) + LM_1(x_{2m}, x_{2m+1}) \\ &\leq \frac{1}{s}(\alpha + \beta) d(x_{2m+1}, x_{2m+2}) \end{aligned}$$

or, $\left(\frac{s-\alpha-\beta}{s}\right) d(x_{2m+1}, x_{2m+2}) \leq 0,$ which is a contradiction.

$$\text{Hence,} \quad x_{2m} = x_{2m+1} \text{ implies } x_{2m+1} = x_{2m+2}. \tag{3.49}$$

For n is odd i.e., for $n = 2m + 1, m \in \mathbb{N} \cup \{0\}$, we can show that

$$x_{2m+1} = x_{2m+2} \text{ implies } x_{2m+2} = x_{2m+3}. \tag{3.50}$$

Hence from (3.49) and (3.50) we have, $x_n = x_{n+1}$ implies $x_{n+1} = x_{n+2}, \; \forall\, n \in \mathbb{N}$.
Proceeding in this manner we have $x_n = x_{n+1}$ implies $x_n = x_{n+k}$ for all $k = 1, 2, ...$
Therefore, $\{x_n\}$ becomes a constant sequence and hence a Cauchy one in X.

Case-II: Assume $x_n \neq x_{n+1}, \ \forall\, n = 1, 2, 3, ...$
Then, for n is even i.e., for $n = 2m$, where $m \in \mathbb{N}$.
Now from (3.45) we have

$$M(x_{2m}, x_{2m-1})$$
$$= \alpha\frac{d(x_{2m}, fx_{2m}) + d(x_{2m}, gx_{2m-1})}{a + d(x_{2m}, x_{2m-1})} + \beta\text{max}\{d(x_{2m}, x_{2m-1}), d(x_{2m-1}, gx_{2m-1}),$$
$$d(x_{2m}, fx_{2m})\} \quad + \gamma\{d(x_{2m}, x_{2m-1}), d(x_{2m-1}, fx_{2m})\}$$
$$= \alpha\frac{d(x_{2m}, fx_{2m+1}) + d(x_{2m-1}, x_{2m})}{a + d(x_{2m}, x_{2m-1})} + \beta\{d(x_{2m}, x_{2m-1}), d(x_{2m-1}, x_{2m}),$$
$$d(x_{2m}, x_{2m+1})\} + \gamma\text{min}\{d(x_{2m}, x_{2m}), d(x_{2m-1}, x_{2m+1})\}$$
$$\leq (\alpha + \beta)d(x_{2m}, x_{2m+1}) \tag{3.51}$$

Now from (3.44) we have

$$\begin{aligned} sd(x_{2m+1}, x_{2m}) &= d(fx_{2m}, gx_{2m-1}) \\ &\leq \xi\{M(x_{2m+2}, x_{2m+1})\}\psi\{M(x_{2m+2}, x_{2m+1})\} \\ &+ LM_1(x_{2m+2}, x_{2m+1}) < \frac{1}{s}\{M(x_{2m}, x_{2m-1})\} + LM_1(x_{2m}, x_{2m-1}) \\ &\leq \frac{1}{s}(\alpha d(x_{2m}, x_{2m+1}) + \beta d(x_{2m}, x_{2m+1})) + Ld(x_{2m}, x_{2m+1}), \end{aligned} \tag{3.52}$$

which gives

$$\left(\frac{s^2 - sL - \alpha - \beta}{s}\right)d(x_{2m}, x_{2m+1}) \leq 0, \qquad \text{which is a contradiction.}$$

Hence, $\qquad d(x_{2m}, x_{2m+1}) < d(x_{2m}, x_{2m-1}).$

Then for $d(x_{2m}, x_{2m+1}) < d(x_{2m-1}, x_{2m})$, from (3.44) we have

$$sd(x_{2m+1}, x_{2m}) \leq \xi\{M(x_{2m}, x_{2m-1})\}\psi\{M(x_{2m}, x_{2m-1})\} + LM_1\{x_{2m}, x_{2m-1}\}$$
$$\text{or,} \quad sd(x_{2m+1}, x_{2m}) < \frac{1}{s}M(x_{2m}, x_{2m-1}) + LM_1M(x_{2m}, x_{2m-1})$$
$$\text{or,} \quad sd(x_{2m+1}, x_{2m}) \leq \frac{1}{s}(\alpha d(x_{2m-1}, x_{2m}) + \beta d(x_{2m-1}, x_{2m})) + Ld(x_{2m}, x_{2m+1})$$
$$\text{or,} \quad \left(\frac{s^2 - sL}{s}\right)d(x_{2m+1}, x_{2m}) < \left(\frac{\alpha + \beta}{s}\right)d(x_{2m-1}, x_{2m})$$
$$\text{or,} \quad \left(\frac{s^2 - sL}{\alpha + \beta}\right)d(x_{2m+1}, x_{2m}) < d(x_{2m-1}, x_{2m})$$
$$\text{or,} \quad d(x_{2m+1}, x_{2m}) < \left(\frac{\alpha + \beta}{s^2 - sL}\right)d(x_{2m-1}, x_{2m}) \tag{3.53}$$

$$\text{Hence,} \qquad d(x_{2m+1}, x_{2m}) < \frac{1}{s}d(x_{2m}, x_{2m-1}) \tag{3.54}$$

For n is odd i.e., for $n = 2m + 1, m \in \mathbb{N} \cup \{0\}$, we have established that

$$d(x_{2m+1}, x_{2m+2}) < \frac{1}{s}d(x_{2m}, x_{2m+1}) \tag{3.55}$$

From (3.54) and (3.55) we have

$$d(x_{n+1}, x_n) < \frac{1}{s}d(x_n, x_{n-1}) \tag{3.56}$$

Claim: $\lim_{n\to\infty} d(x_n, x_{n-1}) = 0.$
As $\{d(x_n, x_{n-1})\}$ is non-negative and monotone decreasing so there exists $l \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n-1}) = l.$$

Now using (3.44) we have

$$\begin{aligned} l = \lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) &\leq \lim_{n\to\infty} sd(f_{2n}, gx_{2n+1}) \\ &\leq \lim_{n\to\infty} \{\xi\{M(x_{2n}, x_{2n+1})\}\psi\{M(x_{2n}, x_{2n+1})\} \\ &+ \lim_{n\to\infty} LM_1(x_{2n}, x_{2n+1}) \\ &\leq \frac{(\alpha+\beta+sL)l}{s} < l, \quad \text{which is a contradiction.} \end{aligned}$$

Hence, $l = 0$. Therefore, $\lim_{n\to\infty} d(x_n, x_{n-1}) = 0$.
Since $\{d(x_n, x_{n-1})\}$ is monotonic decreasing such that $d_n < \frac{1}{s} d_{n-1}$ with $s > 1$ and $\lim_{n\to\infty} d(x_n, x_{n-1}) = 0$, so, by Lemma 3.1 we conclude that $\{x_n\}$ is a Cauchy sequence in X.
Since X is complete, so there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.
Claim: $fz = z$. If not then, $d(fz, z) > 0$.
Now from (3.45) we have

$$\begin{aligned} &M(z, x_{2n+1}) \\ &= \alpha \frac{d(x_{2n+1}, x_{2n+2}) + d(z, fz)}{a + d(x_{2n+1}, z)} + \beta \max\{d(z, x_{2n+1}), d(z, fz), d(x_{2n+1}, x_{2n+2})\} \\ &\quad + \gamma \min\{d(z, x_{2n+2}), d(x_{2n+1}, fz)\} \end{aligned} \tag{3.57}$$

Taking limit as $n \to \infty$ on both sides of (3.57) we have

$$\lim_{n\to\infty} M(z, x_{2n+1}) = (\alpha+\beta) d(fz, z) \tag{3.58}$$

$$\text{and} \quad \lim_{n\to\infty} M_1(z, x_{2n+1}) = 0. \tag{3.59}$$

Now from (3.44) and using (3.58) and (3.59) we have

$$\begin{aligned}
&sd(fz, gx_{2n+1}) \leq \xi\{M(z, x_{2n+1})\}\psi\{M(z, x_{2n+1})\} + LM_1(z, x_{2n+1}) \\
&\text{or,}\;\; sd(fz, gx_{2n+1}) < \frac{1}{s}M(z, x_{2n+1}) + LM_1(z, x_{2n+1}) \\
&\text{or,}\;\; s\lim_{n\to\infty} d(fz, gx_{2n+1}) \leq \frac{1}{s}\lim_{n\to\infty} M(z, x_{2n+1}) + L\lim_{n\to\infty} M_1(z, x_{2n+1}) \\
&\text{or,}\;\; sd(fz, z) \leq \frac{1}{s}(\alpha+\beta)d(z, fz) \\
&\text{or,}\;\; \left(s^2 - \alpha - \beta\right)d(fz, z) \leq 0, \quad \text{which is a contradiction.}
\end{aligned} \tag{3.60}$$

Hence, $fz = z$.

Claim: $gz = z$. If not then $d(gz, z) > 0$.

$$\begin{aligned}
M(x_{2n}, z) =& \alpha\frac{d(x_{2n}, x_{2n+1}) + d(z, gz)}{a + d(x_{2n}, z)} + \beta\max\{d(x_{2n}, z), d(z, gz), d(x_{2n}, x_{2n+1})\} \\
&+ \gamma\min\{d(x_{2n}, gz), d(z, fx_{2n})\}
\end{aligned} \tag{3.61}$$

Taking limit as $n \to \infty$ on both sides of (3.61) we have

$$\lim_{n\to\infty} M(x_{2n}, z) = (\alpha+\beta)d(gz, z) \tag{3.62}$$

$$\text{and} \qquad \lim_{n\to\infty} M_1(z, x_{2n+1}) = 0 \tag{3.63}$$

Now from (3.44) and using (3.62) and (3.63) we have

$$\begin{aligned}
&s\lim_{n\to\infty} d(fz, gx_{2n+1}) \leq \frac{1}{s}\lim_{n\to\infty} M(z, x_{2n+1}) + L\lim_{n\to\infty} M_1(z, x_{2n+1}) \\
&\text{or,}\;\; sd(gz, z) \leq \frac{1}{s}(\alpha+\beta)d(z, gz) \\
&\text{or,}\;\; \left(s^2 - \alpha - \beta\right)d(gz, z) \leq 0, \quad \text{which is a contradiction.}
\end{aligned}$$

Hence, $gz = z$.

Hence, $fz = gz = z$.

Uniqueness: If possible let u, v be two distinct fixed points of f and g. Then

$fu = gu = u, \;\; fv = gv = v$ such that $u \neq v$.
Now

$$M(u,v) = \alpha \frac{d(u,fu)+d(v,gv)}{d(u,v)} + \beta \max\{d(u,v), d(v,gv)+d(u,fu)\} + \gamma \min\{d(u,gu)d(v,fu)\} \leq (\beta+\gamma)d(u,v) \tag{3.64}$$

$$M_1(u,v) = \min\{d(u,v), d(v,gv)+d(u,fu)\} = 0 \tag{3.65}$$

From (3.44) and using (3.64) and (3.65) we have

$$sd(u,v) < \frac{1}{s}[\beta d(u,v) + \gamma d(u,v)]$$
$$\text{or, } \left(\frac{s^2-\beta-\gamma}{s}\right) d(u,v) < 0, \quad \text{which is a contradiction.} \tag{3.66}$$

Therefore, $u = v$. This completes the proof.

The following theorem is on complete partially ordered $b_v(s)$-metric space which is given below: Let X be a non-empty set and a partial order $\preceq$ is defined on X as $x \preceq y$ if and only if $x - y \preceq 0$ and $x \prec y$ if and only if $x - y \prec 0$.

Theorem 3.3. *$(X, \preceq, d)$ be a complete partially ordered $b_v(s)$- metric space with constant $s > 1$. Let $\{T_i\}_{i=1}^{\infty}$ be the sequence of triangular α -admissible non-decreasing self maps with respect to the partial order $\preceq$ on X. Assume that for all $x, y \in X$ with $x \preceq y$, the following hold:*

$$\alpha(x,y)d(T_ix, T_jy) \leq \xi\{M(x,y)\}\psi\{M(x,y)\} + LN(x,y), \qquad \xi \in \Gamma_s; \tag{3.67}$$

where,
$$M(x,y) = a_1 d(x,y) + a_2 \mathit{max}\{d(x,y), d(x,T_ix), d(y,T_jy)\} + a_3 \frac{\mathit{min}\{d(x,T_jy), d(y,T_ix)\}}{1+\mathit{max}\{d(x,T_jy), d(y,T_ix)\}} + a_4\{d(x,y)+d(x,T_ix)+d(y,T_jy)\}; \tag{3.68}$$

and

$$N(x,y) = max\left\{d(x,y), \frac{d(x,T_jy)d(y,T_ix)d(x,y)}{1+d(T_ix,T_jy)d(x,y)}, \frac{d(x,T_ix)d(y,T_jy)}{1+d(x,y)+d(x,T_jy)}\right\}; \tag{3.69}$$

with $a_1+a_2+a_3+3a_4+sL<1; a_i \geq 0,$ *for all* $i=1,2,3,4;$
and
(i) $\alpha(x,x) \geq 1$ *for all* $x \in X$ *and there exists* $x_0 \in X$ *such that* $\alpha(x_0, T_ix_0) \geq 1$ *and* $x_0 \preceq T_ix_0$*; (ii)* $T_i, \ \forall\, i \in \mathbb{N}$ *are continuous mapping. Then* $\{T_i\}_{i=1}^{\infty}$ *have atleast one fixed point* u *in* X.

Also, if v is an another fixed point of $\{T_i\}_{i=1}^{\infty}$ such that u, v are comparable and for two fixed points u, v of $\{T_i\}_{i=1}^{\infty}$ such that $\alpha(u,v) \geq 1$, then $u=v$. i.e., $\{T_i\}_{i=1}^{\infty}$ have a unique common fixed point in X.
Proof: let $x_0 \in X$ be an arbitrary point such that $\alpha(x_0, Tx_0) \geq 1$ and let us define a sequence $\{x_n\} \subset X$ by $x_{n+1} = T_ix_n, \ \ \forall\, i = 1,2,3,...$ and $n \in \mathbb{N}$.
We consider two cases:
Case-I: Assume for some $n \in \mathbb{N},\ x_n = x_{n+1}$ implies $x_{n+1} = x_{n+2}$.
As $x_n = x_{n+1}$, then $T_ix_n = T_ix_{n+1},$ which gives $x_{n+1} = x_{n+2}$.
Proceeding in this manner we can show that if $x_n = x_{n+1},$ then $x_n = x_{n+p},\ p = 1,2,3,...$
Then $\{x_n\}$ becomes a constant sequence and hence a Cauchy one in X.
Case-II: Assume $x_n \neq x_{n+1}, \ \ \forall n \in \mathbb{N}$. Since $x_0 \preceq T_ix_0$ and T_i are non-decreasing mapping, then by induction we have

$$x_0 \preceq T_ix_0 \preceq T_i^2x_0 \preceq ... \preceq T_i^nx_0 \preceq T_i^{n+1} \preceq ...$$

Since T_i are triangular α-admissible, we have

$$\alpha(x_0,x_1) = \alpha(x_0,T_ix_0) \geq 1 \Rightarrow \alpha(T_ix_0,T_ix_1) = \alpha(x_1,x_2) \geq 1.$$

Continuing this process we get,

$$\alpha(x_n,x_{n+1}) \geq 1, \quad \forall n \in \mathbb{N}. \tag{3.70}$$

Claim: $\lim\limits_{n\to\infty} d(x_n,x_{n+1}) = 0.$

Since $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, then by (3.67) and (3.70) we have

$$\begin{aligned} d(x_n, x_{n+1}) =& d(Tx_{n-1}, x_n) \\ & \leq \alpha(x_{n-1}, x_n) d(Tx_{n-1}, x_n) \\ & \leq \xi\{M(x_{n-1}, x_n)\}\psi\{M(x_{n-1}, x_n)\} + LN(x_{n-1}, x_n) \\ & \leq \frac{1}{s}\{M(x_{n-1}, x_n)\} + LN(x_{n-1}, x_n), \end{aligned} \tag{3.71}$$

where,

$$\begin{aligned} & M(x_{n-1}, x_n) \\ & = a_1 d(x_{n-1}, x_n) + a_2 \max\{d(x_{n-1}, x_n), d(x_{n-1}, T_i x_{n-1}), d(x_n, T_j x_n)\} + \\ & \quad a_3 \frac{\min\{d(x_{n-1}, T_j x_n), d(x_n, T_i x_{n-1})\}}{1 + \max\{d(x_{n-1}, T_j x_n), d(x_n, T_i x_{n-1})\}} \\ & + a_4\{d(x_{n-1}, x_n) + d(x_{n-1}, T_i x_{n-1}) + d(x_n, T_j x_n)\}. \\ & = a_1 d(x_{n-1}, x_n) + a_2 \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ & + a_3 \frac{\min\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}}{1 + \max\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}} + a_4\{d(x_{n-1}, x_n) \\ & + d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}. \end{aligned} \tag{3.72}$$

and

$$\begin{aligned} & N(x_{n-1}, x_n) \\ & = \max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, T_j x_n) d(x_n, T_i x_{n-1}) d(x_{n-1}, x_n)}{1 + d(T_i x_{n-1}, T_j x_n) d(x_{n-1}, x_n)}, \right. \\ & \quad \left. \frac{d(x_{n-1}, T_i x_{n-1}) d(x_n, T_j x_n)}{1 + d(x_{n-1}, x_n) + d(x_{n-1}, T_j x_n)}\right\} \\ & = \max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_{n+1}) d(x_n, x_n) d(x_{n-1}, x_n)}{1 + d(x_n, x_{n+1}) d(x_{n-1}, x_n)}, \right. \\ & \quad \left. \frac{d(x_{n-1}, x_n) d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})}\right\} \end{aligned} \tag{3.73}$$

If $d(x_n, x_{n+1}) \geq d(x_{n-1}, x_n)$, then from (3.68) and (3.69) we have

$$M(x_{n-1}, x_n) \leq (a_1 + a_2 + 3a_4) d(x_n, x_{n+1}) \tag{3.74}$$

and

$$N(x_{n-1}, x_n) \leq d(x_n, x_{n+1}) \tag{3.75}$$

Hence from (3.67) and using (3.74), (3.75) we have,

$d(x_n, x_{n+1}) < \frac{(a_1+a_2+3a_4+sL)}{s} d(x_n, x_{n+1})$, which is a contradiction.
Hence, $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$.
Now for $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$, then from (3.68) and (3.69) we have

$$M(x_{n-1}, x_n) \leq (a_1 + a_2 + 3a_4) d(x_{n-1}, x_n) \tag{3.76}$$

and

$$N(x_{n-1}, x_n) \leq d(x_{n-1}, x_n) \tag{3.77}$$

Hence, from (3.71) and using (3.76) and (3.77) we have

$$\begin{aligned} d(x_n, x_{n+1}) &< \frac{(a_1 + a_2 + 3a_4 + sL)}{s} d(x_{n-1}, x_n) \\ \text{or, } d(x_n, x_{n+1}) &< \frac{1}{s} d(x_{n-1}, x_n). \quad [\because (a_1 + a_2 + a_3 + 3a_4) < 1.] \end{aligned} \tag{3.78}$$

Therefore, the sequence $\{d(x_{n-1}, x_n)\}$ is non negative and decreasing and hence convergent.
Let, $\lim_{n\to\infty} d(x_n, x_{n+1}) = l$, where $l \geq 0$.
Now we have

$$\begin{aligned} \frac{1}{s} l \leq l &= \lim_{n\to\infty} d(x_n, x_{n+1}) \\ &\leq \lim_{n\to\infty} \alpha(x_{n-1}, x_n) d(T_i x_{n-1}, T_j x_n) \\ &= \lim_{n\to\infty} \xi\{M(x_{n-1}, x_n)\} \lim_{n\to\infty} \psi\{M(x_{n-1}, x_n)\} + \lim_{n\to\infty} LN(x_{n-1}, x_n) \\ &\leq \frac{(a_1 + a_2 + 3a_4 + sL)}{s} l \\ &< l\frac{1}{s}, \quad \text{which is a contraction.} \end{aligned} \tag{3.79}$$

$$\text{Therefore,} \qquad \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.80}$$

Since $\{d(x_n, x_{n-1})\}$ is monotonic decreasing such that $d_n < \frac{1}{s} d_{n-1}$ with $s > 1$ and $\lim_{n\to\infty} d(x_n, x_{n-1}) = 0$, so, by Lemma 3.1 we conclude that $\{x_n\}$ is a Cauchy

sequence in $(X, \preceq, d)$.
Since X is complete, the sequence $\{x_n\}$ converges to some $u \in X$ i.e., $\lim_{n\to\infty} x_n = u$.
Since T_i are continuous for all $i \in \mathbb{N}$, then we have

$$\lim_{n\to\infty} d(x_{n+1}, T_j u) = \lim_{n\to\infty} d(T_i x_n, T_j u) = d(T_i u, T_j u).$$

Now we show that u is a fixed point of T_i i.e., $d(T_i u, u) = 0$, for all $i = 1, 2, 3, ...$
Now,

$$d(T_i u, u) \leq s[d(T_i u, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+v-1}, x_{n+v}) + d(x_{n+v}, u)] \tag{3.81}$$

Taking limit as $n \to \infty$ on both sides of (87) we get

$$\frac{1}{s} d(T_i u, u) \leq \lim_{n\to\infty} [d(T_i u, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+v-1}, x_{n+v}) + d(x_{n+v}, u)]$$

or,

$$\frac{1}{s} d(T_i u, u) \leq d(T_i u, T_j u) \tag{3.82}$$

and

$$M(u, u) \leq (a_2 + a_3 + 2a_4) d(u, T_i u) \tag{3.83}$$

and

$$N(u, u) \leq d(u, T_i u) \tag{3.84}$$

For $\frac{1}{s} d(T_i u, u) \leq d(T_i u, T_j u)$, it follows from $\alpha(u, u) \geq 1$ that

$$\begin{aligned}
\frac{1}{s} d(T_i u, u) &\leq d(T_i u, T_j u) \leq \alpha(u, u) d(T_i u, T_j u) \leq \xi\{M(u, u)\}\psi\{M(u, u)\} \\
&\quad + LN(u, u) \\
\text{or, } \frac{1}{s} d(T_i u, u) &\leq \xi\{M(u, u)\} M(u, u) + LN(u, u) \\
&\leq \left(\frac{a_2 + a_3 + 2a_4 + sL}{s} \right) d(u, T_i u) \\
&< \frac{1}{s} d(u, T_i u), \qquad \text{which is a contradiction.}
\end{aligned} \tag{3.85}$$

Hence, $T_i u = u$, for all $i = 1, 2, 3, \ldots$

Uniqueness: Assume that u and v be two distinct comparable fixed points of T_i, for all $i = 1, 2, 3, ...$, then $T_i u = u$ and $T_i v = v$, $\forall\, i = 1, 2, 3, ...$
By condition, $\alpha(u, v) \geq 1$.Then

$$\begin{aligned} M(u,v) &\leq a_1 d(u,v) + a_2 d(u,v) + a_3 d(u,v) + a_4 d(u,v) \\ &= (a_1 + a_2 + a_3 + a_4) d(u,v) \end{aligned} \tag{3.86}$$

and

$$N(u,v) \leq a_1 d(u,v) \tag{3.87}$$

$$\begin{aligned} d(u,v) &\leq \alpha(u,v) d(T_i u, T_j v) \leq \xi\{M(u,v)\}\psi\{M(u,v)\} + LN(u,v) \\ &\leq \left(\frac{a_1 + a_2 + a_3 + a_4 + sL}{s}\right) d(u,v) < d(u,v), \quad \text{which is a contradiction.} \end{aligned}$$

Hence, $u = v$.
Therefore $\{T_i\}_{i=1}^{\infty}$ have a unique common fixed point in X.
Note: If $\{T_i\}_{i=1}^{\infty}$ is a sequence of α -admissible non-decreasing self maps with respect to the partial order $\preceq$ on X, the result holds also.

4 Conclusion

$b_v(s)$-metric space is the extension of various metric spaces and in this paper we have established some unique common fixed point theorems on this space. Our results are extensions of various previous results and give some new ideas in this literature.

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