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On the incomplete narayana numbers

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Abstract

In this paper, we first express with sums of binomial coefficients of the Narayana sequence. Moreover, we define the incomplete Narayana numbers and examine their recurrence relations, some properties of these numbers, and the generating function of the incomplete Narayana numbers.

1 Introduction

Filipponi [3] introduced the incomplete Fibonacci and Lucas numbers. The incomplete Fibonacci numbers $F_n(u)$ and Lucas numbers $L_n(v)$ are defined, respectively, by

$$F_n(u) = \sum_{j=0}^{u} \binom{n-1-j}{j}, \quad \left(\lfloor \frac{n-1}{2} \rfloor \leq u \leq n-1\right),$$

and

$$L_n(v) = \sum_{i=0}^{v} \frac{n}{n-i} \binom{n-i}{i}, \quad \left(\lfloor \frac{n}{2} \rfloor \leq v \leq n-1\right).$$

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It is obvious that

$$F_n(\lfloor\frac{n-1}{2}\rfloor) = f_n \quad \text{and} \quad L_n(\lfloor\frac{n}{2}\rfloor) = l_n,$$

where f_n and l_n are the $n-$th Fibonacci and Lucas numbers, respectively.
The generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers were examined by Djordjevic [1]. The incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers were defined by Djordjevic and Srivastava [2]. The generating functions of the incomplete Fibonacci and Lucas numbers were discovered by Pintér and Srivastava [7]. Ramírez [8] presented the bi-periodic incomplete Fibonacci sequences. The incomplete Tribonacci numbers and polynomials were introduced by Ramirez and Sirvent [10]. The incomplete Fibonacci and Lucas $p-$numbers were defined by Tasci and Firengiz [14]. The incomplete bivariate Fibonacci and Lucas $p-$polynomials were defined by Tasci et al. [15].(For other investigations see [9, 11]).

In [5], the Narayana sequence $\{N_r\}_{r\geq 1}$ is defined by

$$N_{r+3} = N_{r+2} + N_r, \quad N_1 = 1, N_2 = 1, N_3 = 1. \tag{1.1}$$

The first few terms of the Narayana numbers are 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189. The Narayana sequence has been the subject of numerous studies, some of which can be observed in references [4–6, 12].

2 The Incomplete Narayana Numbers

In this section, we first give a formula that the Narayana sequence is related to sums of binomial coefficients, Then, we define the incomplete Narayana numbers and obtain some identities for them. Let's begin the study by giving the formula that the Naryana sequence is related to sums of binomial coefficients.

Theorem 2.1. *The Narayana sequence $\{N_r\}_{r\geq 1}$ is related to sums of binomial coefficients by the following identity:*

$$N_r = \sum_{j=0}^{\lfloor\frac{r-1}{3}\rfloor} \binom{r-2j-1}{j}.$$

Proof. By using the method of mathematical induction on r, we aim to establish the validity of the proposition for all natural numbers. For the values of $r = 1, 2, 3, 4, 5, 6$, it is true. Assuma that $r \geq 6$ and the equality holds for r, we have to prove the equality holds for $r + 1$.

Case 1: If $r = 6n$ $(n \geq 1)$, then

$$\begin{aligned} N_{6n+1} &= \sum_{j=0}^{2n} \binom{6n-2j}{j} \\ &= \sum_{j=0}^{2n} \binom{6n-2j-1}{j} + \sum_{j=0}^{2n} \binom{6n-2j-1}{j-1} \\ &= N_{6n} + \sum_{j=-1}^{2n-1} \binom{6n-2j-3}{j} \\ &= N_{6n} + \binom{6n-1}{-1} + \sum_{j=0}^{2n-1} \binom{6n-2j-3}{j} \\ &= N_{6n} + N_{6n-2}. \end{aligned}$$

Case 2: If $r = 6n + 1$ $(n \geq 1)$, then

$$\begin{aligned} N_{6n+2} &= \sum_{j=0}^{2n} \binom{6n-2j+1}{j} \\ &= \sum_{j=0}^{2n} \binom{6n-2j}{j} + \sum_{j=0}^{2n} \binom{6n-2j}{j-1} \\ &= N_{6n+1} + \binom{6n}{-1} - \binom{2n-2}{2n} + \sum_{j=0}^{2n} \binom{6n-2j-2}{j} \\ &= N_{6n+1} + N_{6n-1}. \end{aligned}$$

Case 3: If $r = 6n+2$ $(n \geq 1)$, then

$$\begin{aligned}
N_{6n+3} &= \sum_{j=0}^{2n} \binom{6n-2j+2}{j} \\
&= \sum_{j=0}^{2n} \binom{6n-2j+1}{j} + \sum_{j=0}^{2n} \binom{6n-2j+1}{j-1} \\
&= N_{6n+2} + \sum_{j=-1}^{2n-1} \binom{6n-2j-1}{j} \\
&= N_{6n+2} + \binom{6n+1}{-1} - \binom{2n-1}{2n} + \sum_{j=0}^{2n} \binom{6n-2j-1}{j} \\
&= N_{6n+2} + N_{6n}.
\end{aligned}$$

Case 4: If $r = 6n+3$ $(n \geq 1)$, then

$$\begin{aligned}
N_{6n+4} &= \sum_{j=0}^{2n+1} \binom{6n-2j+3}{j} \\
&= \sum_{j=0}^{2n+1} \binom{6n-2j+2}{j} + \sum_{j=0}^{2n+1} \binom{6n-2j+2}{j-1} \\
&= \binom{2n}{2n+1} + \sum_{j=0}^{2n} \binom{6n-2j+2}{j} + \sum_{j=-1}^{2n} \binom{6n-2j}{j} \\
&= N_{6n+3} + \binom{6n+2}{-1} + \sum_{j=0}^{2n} \binom{6n-2j}{j} \\
&= N_{6n+3} + N_{6n+1}.
\end{aligned}$$

Case 5: If $r = 6n + 4$ $(n \geq 1)$, then

$$\begin{aligned} N_{6n+5} &= \sum_{j=0}^{2n+1} \binom{6n-2j+4}{j} \\ &= \sum_{j=0}^{2n+1} \binom{6n-2j+3}{j} + \sum_{j=0}^{2n+1} \binom{6n-2j+3}{j-1} \\ &= N_{6n+4} + \sum_{j=-1}^{2n} \binom{6n-2j+1}{j} \\ &= N_{6n+4} + \binom{6n+3}{-1} + \sum_{j=0}^{2n} \binom{6n-2j+1}{j} \\ &= N_{6n+4} + N_{6n+2}. \end{aligned}$$

Case 6: If $r = 6n + 5$ $(n \geq 1)$, then

$$\begin{aligned} N_{6n+6} &= \sum_{j=0}^{2n+1} \binom{6n-2j+5}{j} \\ &= \sum_{j=0}^{2n+1} \binom{6n-2j+4}{j} + \sum_{j=0}^{2n+1} \binom{6n-2j+4}{j-1} \\ &= N_{6n+5} + \sum_{j=-1}^{2n} \binom{6n-2j+2}{j} \\ &= N_{6n+5} + \binom{6n+4}{-1} + \sum_{j=0}^{2n} \binom{6n-2j+2}{j} \\ &= N_{6n+5} + N_{6n+3}. \end{aligned}$$

□

Definition 2.1. *The incomplete Narayana numbers $N_r(y)$ are defined by*

$$N_r(y) = \sum_{j=0}^{y} \binom{r-2j-1}{j}, \quad \left(r = 1, 2, 3, ...; 0 \leq y \leq \lfloor \frac{r-1}{3} \rfloor = \hat{r}\right). \quad (2.1)$$

The numbers $N_r(y)$ are displayed in Table 1 for the first few values of r and the related permissible values of y.

Table 1: The first few values of the incomplete Narayana Numbers

$r \setminus y$	0	1	2	3	4
1	1				
2	1				
3	1				
4	1	2			
5	1	3			
6	1	4			
7	1	5	6		
8	1	6	9		
9	1	7	13		
10	1	8	18	19	
11	1	9	24	27	
12	1	10	31	40	

The relation (2.1) has some special cases as following:

- $$N_r(0) = 1, \quad (r \geq 1)$$
- $$N_r(1) = r - 2, \quad (r \geq 4)$$
- $$N_r(2) = \frac{r^2 - 9r + 26}{2}, \quad (r \geq 7)$$
- $$N_r(\hat{r}) = N_r, \quad (r \geq 1).$$

2.1 Some Identities of the Incomplete Narayana Numbers $N_r(y)$

Proposition 2.1. *The incomplete Narayana numbers $N_r(y)$ can be given by the recurrence relation*

$$N_{r+3}(y+1) = N_{r+2}(y+1) + N_r(y), \quad 0 \leq y \leq \hat{r}. \tag{2.2}$$

Proof. Using the Definition (2.1), we obtain the desired equality as follows:

$$\begin{aligned} N_{r+2}(y+1) + N_r(y) &= \sum_{j=0}^{y+1} \binom{r-2j+1}{j} + \sum_{j=0}^{y} \binom{r-2j-1}{j} \\ &= \sum_{j=0}^{y+1} \binom{r-2j+1}{j} + \sum_{j=1}^{y+1} \binom{r-2j+1}{j-1} + \binom{r+1}{-1} \\ &= \sum_{j=0}^{y+1} \binom{r-2j+1}{j} + \sum_{j=0}^{y+1} \binom{r-2j+1}{j-1} \\ &= \sum_{j=0}^{y+1} \binom{r-2j+2}{j}. \end{aligned}$$

Hence, we obtain

$$N_{r+2}(y+1) + N_r(y) = N_{r+3}(y+1).$$

□

Proposition 2.2. *The following identity holds:*

$$N_{r+3}(y) = N_{r+2}(y) + N_r(y) - \binom{r-2y-1}{y} \tag{2.3}$$

Proof. It is clear that

$$\begin{aligned}
N_{r+3}(y) &= \sum_{j=0}^{y} \binom{r-2j+2}{j} \\
&= \sum_{j=0}^{y} \left[\binom{r-2j+1}{j} + \binom{r-2j+1}{j-1} \right] \\
&= \sum_{j=0}^{y} \binom{r-2j+1}{j} + \sum_{j=0}^{y} \binom{r-2j+1}{j-1} \\
&= N_{r+2}(y) + \sum_{j=-1}^{y-1} \binom{r-2j-1}{j} \\
&= N_{r+2}(y) + \binom{r+1}{-1} + \sum_{j=0}^{y} \binom{r-2j-1}{j} - \binom{r-2y-1}{y}.
\end{aligned}$$

Thus, we have

$$N_{r+3}(y) = N_{r+2}(y) + N_r(y) - \binom{r-2y-1}{y}.$$

This completes the proof. □

Proposition 2.3. *The following identity holds:*

$$\sum_{a=0}^{n} N_{r+2a}(y+a) \binom{n}{a} = N_{r+3n}(y+n), \quad 0 \leq k \leq \frac{n-h-1}{2}. \tag{2.4}$$

Proof. We proceed by induction on n. The sum (2.4) plainly valid for $n = 1$. Suppose it holds for a specific $n > 1$. For the inductive step $n \to n+1$, we have

$$\begin{aligned}
N_{r+3n+3}(y+n+1) &= \sum_{a=0}^{n+1} N_{r+2a}(y+a) \binom{n+1}{a} \\
&= \sum_{a=0}^{n+1} N_{r+2a}(y+a) \binom{n}{a} + \sum_{a=0}^{n+1} N_{r+2a}(y+a) \binom{n}{a-1}
\end{aligned}$$

$$= \sum_{a=0}^{n} N_{r+2a}(y+a)\binom{n}{a} + \binom{n}{n+1} N_{r+2n+2}(y+n+1)$$
$$+ \sum_{a=0}^{n} N_{r+2a+2}(y+a+1)\binom{n}{a} + \binom{n}{-1} N_r(y).$$

Thus, we prove

$$N_{r+3n+3}(y+n+1) = N_{r+3n}(y+n) + N_{r+3n+2}(y+n+1).$$

This completes the proof. □

Proposition 2.4. *For $n \geq 2y+3$, we have*

$$\sum_{a=0}^{h-1} N_{r+a}(y) = N_{r+h+2}(y+1) - N_{r+2}(y+1). \tag{2.5}$$

Proof. Using equality (2.2) repeatedly,

$$N_{r+3}(y+1) = N_{r+2}(y+1) + N_r(y)$$

$$N_{r+4}(y+1) = N_{r+3}(y+1) + N_{r+1}(y)$$
$$\cdots$$
$$N_{r+h+1}(y+1) = N_{r+h}(y+1) + N_{r+h-2}(y)$$

$$N_{r+h+2}(y+1) = N_{r+h+1}(y+1) + N_{r+h-1}(y).$$

We get

$$\sum_{a=0}^{h-1} N_{r+a}(y) = N_{r+h+2}(y+1) - N_{r+2}(y+1).$$

□

3 Generating Functions of the Incomplete Narayana Numbers

In this section, we give the generating functions of the incomplete Narayana numbers.

Lemma 3.1. *Let $\{S_r\}_{r=0}^{\infty}$ be a complex sequence satisfying the following nonhomogeneous recurrence relation:*

$$S_r = S_{r-1} + S_{r-3} + m_r, \quad (r > 3)$$

where $\{m_r\}$ is given a complex sequence. Then the generating function $G_S(t)$ of the sequence $\{S_r\}$ is

$$G_S(t) = \frac{S_0 - m_0 + (S_1 - S_0 - m_1)t + (S_2 - S_1 - m_2)t^2 + G(t)}{1 - t - t^3}$$

where $G(t)$ denotes the generating function of $\{m_r\}$.

Proof. Let

$$G_S(t) = \sum_{r=0}^{\infty} S_r t^r = S_0 + S_1 t + S_2 t^2 + S_3 t^3 + \cdots + S_r t^r + \ldots.$$

Multiply the above equation by $-t$ and $-t^3$, respectively.

$$-tG_S(t) = -S_0 t - S_1 t^2 - S_2 t^3 - S_3 t^4 - \cdots - S_{r-1} t^r + \ldots$$
$$-t^3 G_S(t) = -S_0 t^3 - S_1 t^4 - S_2 t^5 - S_3 t^6 - \cdots - S_{r-3} t^r + \ldots$$

and

$$G(t) = m_0 + m_1 t + m_2 t^2 + m_3 t^3 + \cdots + m_r t^r + \ldots$$

Therefore, we get

$$G_S(t)(1 - t - t^3) - G(t) = S_0 - m_0 + (S_1 - S_0 - m_1)t + (S_2 - S_1 - m_2)t^2.$$

□

Theorem 3.1. *The generating function of the incomplete Narayana numbers* $N_r(y)$ *is given by*

$$G_{N_r}(t) = \sum_{n=0}^{\infty} N_r(y)t^r = \frac{N_{3y+1} + N_{3y-2}t + N_{3y}t^2 + \frac{t+3}{(1-t)^{y+1}}}{1 - t - t^3}.$$

Proof. Assume that y is a fixed positive integer. From (2.1) and (2.3), $N_r(y) = 0$ for $0 \leq n < 2k$, $N_{3y+1}(y) = N_{3y+1}$, $N_{3y+2}(y) = N_{3y+2}$ and $N_{3y+3}(y) = N_{3y+3}$ and that

$$N_r(y) = N_{r-1}(y) + N_{r-3}(y) - \binom{r-2y-4}{r-3y-4}.$$

Now let $S_0 = N_{3y+1}(y)$, $S_1 = N_{3y+2}(y)$, $S_2 = N_{3y+3}(y)$ and $S_r = N_{3y+r+1}(y)$. Also let $m_0 = m_1 = m_2 = 0$ and

$$m_r = \binom{r-3+y}{r-3}.$$

Note: (From [13], p. 355, Equation 7.1(5)), it is know that the equality holds

$$\sum_{r=0}^{\infty} \binom{\alpha + (\beta+1)r}{r} t^r = \frac{(1+\lambda)^{\alpha+1}}{1-\beta\lambda},$$

where α and β are complex numbers independent of r, and λ is a function of t defined implicitly by

$$\lambda = t(1+\lambda)^{\beta+1}, \quad \lambda(0) = 0.$$

Then, the generating function of the sequence $\{m_r\}$ is

$$G(t) = \frac{t^3}{(1-t)^{y+1}}$$

(see [13]). Thus, from Lemma 3.1, we get the generating function $G_{N_r}(t)$ of sequence $\{S_r\}$. □

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