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Numerical solution of linear Emden Fowler equations using collocation method

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Abstract

In this paper, we introduce a new collocation approach for the numerical solution of linear Emden Fowler equations. The problems is first written in the integral form and latter converted to system of linear equations using standard collocation points. Test problems solved by iterative methods are used to test the performances of the new method which is found to be accurate and simple to implement.

1 Introduction

Lane Emden equation was first discovered during a research on thermal behavior of spherical cloud of gas acting under the mutual attraction of its molecules subject to the classical law of thermodynamics [1]. Since then, it has been applied to several phenomena in mathematical physics and astrophysics including theory of stellar structure, isothermal gas spheres, equilibrium density distribution in self gravitating sphere of isothermal gas and other areas of energy transport [2, 3].

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Many methods developed in literature for the solution of Emden Fowler equations include: Adomian decomposition method [4, 5, 6], Variational iterative method [7, 8, 9, 10], Pade approximate method [11] which are all iterative methods. Other methods include: Collocation method [12, 13, 14, 15, 16], Fuzzy method [17] and Neural computing [18].

The advantages of this new method over other collocation methods mentioned above is in it's simplicity in development and implementation, with better accuracy. Moreover, this new method is not restricted to a particular method of choosing collocation points.

In this study, we propose a new method for the solution of the equation

$$u''(t) = g(t) + \frac{h}{t}u'(t) + p(t)\,u(t)\,,\ t \in [0,1] \tag{1.1}$$

subject to the initial condition

$$u(0) = \gamma_0,\ u'(0) = \gamma_1, \tag{1.2}$$

where p, $g : [0,1] \to \mathbb{R}$ are continuous functions, h, γ_0, $\gamma_1 \in \mathbb{R}$, $u : [0,1] \to \mathbb{R}$ is the solution to be determined. If $P(t) = 1$, (1.1) is referred to as Lane Emden's equation.

The remaing part of the paper is organised as follows: In the Section 2, we give basic definitions and results that will be used in this study. In Section 3, we discuss the derivation of the new method, convergence of method of solution and implementation. Results are presented in Section 4 and we conclude in Section 5.

2 Preliminary Study

In the course of the development of the new method, we will use the following definitions and results.

Lemma 2.1. *[19] The Riemann Liouville integral operator $I^n(.)$, where n is the order, on a usual Lesbgue space is defined as*

$$_0I_t^n(u(t)) = \frac{1}{\Gamma(n)}\int_0^t (t-s)^{n-1}\,u(s)\,ds \tag{2.1}$$

Lemma 2.2. *[19] Let $y(t)$ be a continuous function, $I^n(.)$ and $D^n(.)$ are the* *integral and differential operators of order n respectively, then*

$$
{}_0I_x^n\left({}_0^cD_x^n\left(s(x)\right)\right) = s(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} s^{(k)}(0) \tag{2.2}
$$

Lemma 2.3. *Let $\alpha, \beta \in \mathbb{R}^+$, then*

$$
\int_0^x (x-t)^{\alpha-1} t^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} x^{\alpha+\beta-1} \tag{2.3}
$$

Definition 2.1. *[20] (fixed point) Let (X, d) be a metrix space, a fixed point of a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself, that is $Tx = x$*

3 Methodology

3.1 Method of solution

Equation (1.1) is first converted into integral equation by multiplying it by ${}_0I_x^2$, applying (2.1) and (2.2) ,we get

$$
u\left(t\right) = v\left(t\right) + h \int_0^t \left(t-s\right) \frac{1}{s} u'\left(s\right) ds + \int_0^t \left(t-s\right) p\left(s\right) u\left(s\right) ds \tag{3.1}
$$

where $v\left(t\right) = \gamma_0 + \gamma_1 t + \int_0^t \left(t-s\right) g\left(s\right) ds$. We assume the approximating solution of (3.1) in the form

$$
u_N\left(t\right) = \sum_{n=0}^{1} a_n \cos\left(nt\right) + \sum_{n=2}^{N} a_n t^n, \; N \in \mathbb{Z}^+ \tag{3.2}
$$

so that the first term of the RHS of (3.2) will effectively overcome the point of singularity. Expressing (3.2) matrix form yield

$$
u\left(t\right) = \mathbf{T}\left(t\right)\mathbf{A} \tag{3.3}
$$

where $\mathbf{T}(t) = \begin{bmatrix} 1 & \eta_0(t) & t^2 & \cdots & t^N \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T$ are constants to be determined. $\eta_0(t) = \sum_{j=0}^{J} \dfrac{(-1)^j t^{2j}}{\Gamma(2j+1)}, j \in \mathbb{Z}^+$

$$u'(t) = \mathbf{T}'(t)\,\mathbf{A} \tag{3.4}$$

where $\mathbf{T}'(t) = \begin{bmatrix} 0 & \eta_0'(t) & t & \cdots & t^{N-1} \end{bmatrix}$, $\eta_0'(t) = \sum_{j=0}^{J} \dfrac{(-1)^j (2j)\, t^{2j-1}}{\Gamma(2j+1)}$.

Using (3.2) and (3.4) in (3.1)

$$\begin{aligned} \mathbf{T}(t)\,\mathbf{A} = \quad & v(t) + h\int_0^t (t-s)\frac{1}{s}\begin{bmatrix} 0 & \eta_0'(s) & s & \cdots & s^{N-1} \end{bmatrix} ds\mathbf{A} \\ & + \int_0^t (t-s)\,P(s)\begin{bmatrix} 1 & \eta_0(s) & s^2 & \cdots & s^N \end{bmatrix} ds\mathbf{A}, \end{aligned} \tag{3.5}$$

we again write $P(s) = \sum_{r=0}^{R} p_r s^r$ and solving (3.5) by using (2.2) and (2.3) , we obtain

$$\left(I_0(t) - hI_1(t) - \sum_{r=0}^{R} I_2(t) \right) \mathbf{A} = v(t), \tag{3.6}$$

where $I_1(t) = [0\ \theta_1(t)\ \ \alpha_1(t;2)\ \cdots\ \alpha_1(t;N)]$, $\theta_1(t)$
$= \sum_{j=0}^{J} \dfrac{(-1)^{j+1}}{(2j+1)\,\Gamma(2j+3)} t^{2j+2}$, $\alpha_1(t;n)$
$= \dfrac{1}{(n-1)} t^n$, $n = 0(1)N$, $I_2(t,r) = [\alpha_2(t;0)\ \ \theta_2(t)\ \ \alpha_2(t;2)\ \cdots\ \alpha_2(t;N)]$,
$\theta_2(t)$
$= \sum_{j=0}^{J}\sum_{r=0}^{R} \dfrac{p_r(-1)^j\,\Gamma(2j+r+1)}{\Gamma(2j+1)\,\Gamma(2j+r+3)} t^{2j+r+2}$, $\alpha_2(t;n)$
$= \sum_{r=0}^{R} \dfrac{p_r\Gamma(n+r+1)}{\Gamma(n+r+3)}$
t^{n+r+2}.

Collocating (3.6) using the standard collocation points $t_j = a + \dfrac{b-a}{N} j$, $[a,b] = [0,1]$, $j = 0, 1, \cdots N$ and solving for $\mathbf{A}$ using matrix inversion method gives the desired approximate solution

$$u_N(t) = \mathbf{I}_0(t)\,\mathbf{w}(t_j)^{-1}\,\mathbf{v}(t_j), \tag{3.7}$$

where $\mathbf{w}(t_j) = \mathbf{I}_0(t_j) - h\mathbf{I}_1(t_j) - \mathbf{I}_2(t_j)$, $\mathbf{v}(t_j) = \begin{bmatrix} v(t_0) & v(t_1) & \cdots & v(t_N) \end{bmatrix}^T$,

$$\mathbf{I}_n(t_j) = \begin{bmatrix} I_n(t_0) & I_n(t_1) & \cdots & I_n(t_N) \end{bmatrix}^T, n = 0, 1, 2$$

3.2 Convergence of solution

Lemma 3.1. *Let (X, d) be a metrix space and $T : X \to X$ be a mapping, If $u_N(t)$ and $u_{N-1}(t) \in X$ are convergent approximate solutions, then*

$$\lim_{N\to\infty} (u_N(t) - \lim u_{N-1}(t)) = 0 \tag{3.8}$$

Proof. Let $u(t) \in X$ be the exact solution, since $u_N(t)$ and $u_{N-1}(t)$ are convergent, then

$$\lim_{N\to\infty} u_N(t) = u(t)$$

Moreover

$$\lim_{N\to\infty} u_{N-1}(t) = u(t)$$

hence

$$\lim_{N\to\infty} (u_N(t) - \lim u_{N-1}(t)) = 0$$

which implies that the solution converges to a unique fixed point in X. □

Theorem 3.1. *(convergence) Let (X, d) be a metrix space and $T : X \to X$ be a continuous mapping, If $u_N(t)$ and $u_{N-1}(t) \in X$ are approximate solutions of* (3.1) *such that*

$$\lim_{N\to\infty} (Tu_N(t) - Tu_{N-1}(t)) = 0$$

then, the approximate solution converges to the exact solution.

Proof. Using the fixed point theorem on (3.1), then

$$Tu_N(t) = v(t) + h\int_0^t (t-s)\frac{1}{s}u_N'(s)\,ds + \int_0^t (t-s)\,p(s)\,u_N(s)\,ds \tag{3.9}$$

the approximate solution can be written as

$$u_N = \sum_{n=0}^{1}\sum_{j=0}^{J} \frac{a_n(-1)^j n^{2j}}{\Gamma(2j+1)} s^{2j} + \sum_{n=2}^{N} a_n s^n \tag{3.10}$$

$$u'_N = \sum_{n=0}^{1}\sum_{j=0}^{J}\frac{a_n\left(-1\right)^j\left(2j\right)n^{2j}}{\Gamma\left(2j+1\right)}s^{2j-1} + \sum_{n=2}^{N} na_n s^{n-1} \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9) gives

$$\begin{aligned} Tu_N\left(t\right) = v\left(t\right) + h\sum_{n=0}^{1}\sum_{j=0}^{J}\frac{a_n\left(-1\right)^j n^{2j}}{\Gamma\left(2j+1\right)}\int_0^t \left(t-s\right)s^{2j-2}ds \\ + \sum_{n=2}^{N} na_n s^{n-1}\int_0^t \left(t-s\right)s^{n-2}ds \\ + \sum_{n=0}^{1}\sum_{j=0}^{J}\frac{a_n\left(-1\right)^j n^{2j}}{\Gamma\left(2j+1\right)}\int_0^t \left(t-s\right)p\left(s\right)s^{2j}ds \\ + \sum_{n=2}^{N} a_n \int_0^t \left(t-s\right)p\left(s\right)s^{n}ds \end{aligned} \tag{3.12}$$

writing $p\left(s\right) = \sum_{r=0}^{R} p_r s^r$, hence

$$\begin{aligned} Tu_N\left(t\right) = v\left(t\right) + h\sum_{n=0}^{1}\sum_{j=0}^{J}\frac{a_n\left(-1\right)^j n^{2j}}{\Gamma\left(2j+1\right)}\int_0^t \left(t-s\right)s^{2j-2}ds \\ + \sum_{n=2}^{N} na_n \int_0^t \left(t-s\right)s^{n-2}ds \\ + \sum_{n=0}^{1}\sum_{j=0}^{J}\sum_{r=0}^{R}\frac{a_n p_r\left(-1\right)^j n^{2j}}{\Gamma\left(2j+1\right)}\int_0^t \left(t-s\right)p\left(s\right)s^{2j+r}ds \\ + \sum_{n=2}^{N}\sum_{r=0}^{R} p_r a_n \int_0^t \left(t-s\right)p\left(s\right)s^{n+r}ds \end{aligned} \tag{3.13}$$

Applying (2.3) on (3.13)

$$(Tu_N)(t) = v(t) + h\sum_{n=0}^{1}\sum_{j=0}^{J}\frac{a_n(-1)^j n^{2j}\Gamma(2)\Gamma(2j-1)}{(\Gamma(2j+1))^2}t^{2j}$$
$$+\sum_{n=2}^{N}\frac{na_n\Gamma(2)\Gamma(n-1)}{\Gamma(n+1)}t^n$$
$$+\sum_{n=0}^{1}\sum_{j=0}^{J}\sum_{r=0}^{R}\frac{a_n p_r(-1)^j n^{2j}\Gamma(2)\Gamma(2j+r+1)}{\Gamma(2j+1)\Gamma(2j+r+3)}t^{2j+r+2}$$
$$+\sum_{n=2}^{N}\sum_{r=0}^{R}\frac{p_r a_n\Gamma(2)\Gamma(n+r+1)}{\Gamma(n+r+3)}t^{n+r+2} \tag{3.14}$$

Let $u_{N-1}(t) = \sum_{n=0}^{1} a_n\cos(nt) + \sum_{n=2}^{N-1} b_n t^n$, $N \in \mathbb{Z}^+$

$$(Tu_{N-1})(t) = v(t) + h\sum_{n=0}^{1}\sum_{j=0}^{J}\frac{b_n(-1)^j n^{2j}\Gamma(2)\Gamma(2j-1)}{(\Gamma(2j+1))^2}t^{2j}$$
$$+\sum_{n=2}^{N-1}\frac{nb_n\Gamma(2)\Gamma(n-1)}{\Gamma(n+1)}t^n$$
$$+\sum_{n=0}^{1}\sum_{j=0}^{J}\sum_{r=0}^{R}\frac{b_n p_r(-1)^j n^{2j}\Gamma(2)\Gamma(2j+r+1)}{\Gamma(2j+1)\Gamma(2j+r+3)}t^{2j+r+2}$$
$$+\sum_{n=2}^{N-1}\sum_{r=0}^{R}\frac{p_r b_n\Gamma(2)\Gamma(n+r+1)}{\Gamma(n+r+3)}t^{n+r+2} \tag{3.15}$$

$$
\begin{aligned}
&\left|\left(Tu_N\right)(t)-\left(Tu_{N-1}\right)(t)\right| \\
&=h\sum_{n=0}^{1}\sum_{j=0}^{J}\frac{(-1)^j n^{2j}\Gamma(2)\Gamma(2j-1)}{\left(\Gamma(2j+1)\right)^2}\left|a_n-b_n\right|t^{2j} \\
&+\frac{Na_N\Gamma(2)\Gamma(N-1)}{\Gamma(N+1)}t^N+\sum_{n=2}^{N-1}\frac{n\Gamma(2)\Gamma(n-1)}{\Gamma(n+1)}\left|a_n-b_n\right|t^n \qquad (3.16)\\
&+\sum_{n=0}^{1}\sum_{j=0}^{J}\sum_{r=0}^{R}\frac{p_r(-1)^j n^{2j}\Gamma(2)\Gamma(2j+r+1)}{\Gamma(2j+1)\Gamma(2j+r+3)}\left|a_n-b_n\right|t^{2j+r+2} \\
&+\sum_{r=0}^{R}\frac{p_r a_N\Gamma(2)\Gamma(N+r+1)}{\Gamma(N+r+3)}t^{N+r+2} \qquad (3.17)\\
&+\sum_{n=2}^{N-1}\sum_{r=0}^{R}\frac{p_r\Gamma(2)\Gamma(n+r+1)}{\Gamma(n+r+3)}\left|a_n-b_n\right|t^{n+r+2} \qquad (3.18)
\end{aligned}
$$

since $t\in[0,1]\,,|a_n-b_n|\neq 0$, hence obviousely

$$\lim_{N\to\infty}\left(Tu_N(t)-Tu_{N-1}(t)\right)\to 0,\ \forall r$$

Therefore the method converges to the exact solution. □

4 Numerical Examples

In this section, we solve examples to test the efficiency of the new method. All computations in this section are done with the aid of program written in MATLAB (2015a) and run on a PC.

Example 4.1. *[7] Consider*

$$u''(t)+\frac{8}{t}u'(t)+tu(t)=t^5-t^4+44t^2-30t \qquad (4.1)$$

$0\leq t\leq 1$, *with* $u(0)=0,\ u'(0)=0$.*The solution* $u(t)=t^4-t^3$. *Comparing with* (1.1), $g(t)=t^5-t^4+44t^2-30t$, $h=8$, $P(t)=t$. *The integral form gives*

$$
\begin{aligned}
u(t)+8\int_0^t(t-s)\frac{1}{s}u'(s)\,ds+\int_0^t(t-s)\,su(s)\,ds \\
=\int_0^t(t-s)\left(s^5-s^4+44s^2-30s\right)ds
\end{aligned}
$$

Taking $N = 4$ *and* $J = 1$, $r = 1$, $I_0(t) = \begin{bmatrix} 1 & 1 & t^2 & 0.5t^3 & 0.3333t^4 \end{bmatrix}$, *collocating using the standard points* $t_j = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix}$, *hence,*

$$\mathbf{I}_0(t_j) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} \\ 1 & 1 & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ 1 & 1 & \frac{9}{18} & \frac{27}{64} & \frac{81}{256} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \mathbf{v}(t_j)$$
$$= \begin{bmatrix} 0 & \frac{-73181}{1146880} & \frac{-10649}{26850} & \frac{-1091799}{1146880} & \frac{-47}{35} \end{bmatrix}^T.$$

$$I_2(t) = \begin{bmatrix} \frac{t^3}{6} & \frac{t^3}{6} & \frac{t^5}{20} & \frac{t^6}{30} & \frac{t^7}{42} \end{bmatrix}, \text{ then,}$$
$$\mathbf{I}_2(t_j) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{384} & \frac{1}{384} & \frac{1}{20480} & \frac{1}{122880} & \frac{1}{1688128} \\ \frac{1}{48} & \frac{1}{48} & \frac{1}{640} & \frac{1}{1920} & \frac{1}{5376} \\ \frac{9}{128} & \frac{9}{128} & \frac{243}{20480} & \frac{243}{40960} & \frac{72}{229376} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{20} & \frac{1}{30} & \frac{1}{42} \end{bmatrix}.$$

$$I_1(t) = \begin{bmatrix} 0 & \frac{-t^2}{2} & \frac{t^2}{2} & \frac{t^3}{2} & \frac{t^4}{3} \end{bmatrix}, \text{hence, } \mathbf{I}_1(t_j) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{32} & \frac{1}{16} & \frac{1}{128} & \frac{1}{768} \\ 0 & \frac{-1}{8} & \frac{1}{4} & \frac{1}{16} & \frac{1}{48} \\ 0 & \frac{-9}{32} & \frac{9}{16} & \frac{27}{128} & \frac{27}{256} \\ 0 & \frac{-1}{2} & 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}.$$

Therefore $\mathbf{w}(t_j) = \mathbf{I}_0(t_j) + 8\mathbf{I}_1(t_j) + \mathbf{I}_2(t_j)$,

$$\mathbf{w}(t_j) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ \frac{385}{384} & \frac{289}{384} & \frac{11521}{20480} & \frac{9601}{122880} & \frac{9857}{688128} \\ \frac{49}{48} & \frac{1}{48} & \frac{1441}{640} & \frac{1201}{1920} & \frac{411}{1792} \\ \frac{137}{128} & \frac{-151}{128} & \frac{103923}{20480} & \frac{86643}{40960} & \frac{266841}{229376} \\ \frac{7}{6} & \frac{-17}{6} & \frac{181}{20} & \frac{151}{30} & \frac{155}{421} \end{bmatrix}$$

Solving for $\mathbf{A} = \mathbf{w}(t_j)^{-1}\mathbf{v}(t_j)$, gives $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \end{bmatrix}^T$. Substituting back into the approximate solution gives $u(t) = t^4 - t^3$, which is the exact solution. Same results are obtained when $N > 4$.

Example 4.2. *[21] Consider*

$$u''(t) + \frac{2}{t}u'(t) + u^m(t) = 0,\ m = 0 \text{ or } 1,\ u(0) = 1,\ u'(0) = 0 \tag{4.2}$$

The solution $u(t) = 1 - \dfrac{t^2}{2}$*, when* $m = 0$ *and* $u(t) = \dfrac{\sin(t)}{t}$ *when* $m = 1$*. We write the solution in the form* $u(N; J)$*.*

If $m = 1$, using $N = 5$, $J = 3$, $r = 0$, for illustration, following the steps in Example 1, we obtain the following: $2I_2(t) = \begin{bmatrix} 0 & \theta_2(t) & t^2 & \frac{t^3}{2} & \frac{t^4}{3} & \frac{t^5}{5} \end{bmatrix}$, $\theta_2(t) = \dfrac{t^6}{3600} - \dfrac{t^4}{72} + \dfrac{t^2}{2}$, $I_1(t) = \begin{bmatrix} \frac{t^2}{2} & \theta_1(t) & \frac{t^4}{12} & \frac{t^5}{20} & \frac{t^6}{30} & \frac{t^7}{42} \end{bmatrix}$, $\theta_1(t) = \dfrac{t^6}{720} - \dfrac{t^4}{24} + \dfrac{t^2}{2}$, $I_0(t) = \begin{bmatrix} 1 & \eta_0(t) & t^2 & t^3 & t^4 & t^5 \end{bmatrix}$, $\mathbf{v}(t_j) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$. $\eta_0(t) = \dfrac{t^4}{24} - \dfrac{t^2}{2} + 1$, $\mathbf{w}(t_j) = \mathbf{I}_0(t_j) + 2\mathbf{I}_1(t_j) + \mathbf{I}_2(t_j)$, $\mathbf{w}(t_j) =$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{51}{50} & \frac{54002503}{56250000} & \frac{901}{7500} & \frac{1001}{62500} & \frac{417}{156250} & \frac{788}{1640625} \\ \frac{27}{25} & \frac{2955637}{3515625} & \frac{904}{1875} & \frac{2008}{15625} & \frac{3344}{78125} & \frac{25264}{1640625} \\ \frac{59}{50} & \frac{4022743}{6250000} & \frac{2727}{2500} & \frac{27243}{62500} & \frac{33993}{156250} & \frac{64152}{546875} \\ \frac{33}{25} & \frac{1306393}{3515625} & \frac{3664}{1875} & \frac{16256}{15625} & \frac{54016}{78125} & \frac{814592}{1640625} \\ \frac{3}{2} & \frac{103}{3600} & \frac{37}{12} & \frac{41}{20} & \frac{17}{10} & \frac{32}{21} \end{bmatrix},$$

therefore $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{-1}{80} & 0 \end{bmatrix}^T$. Substituting back into the approximate solution gives

$$u(5,3) = 1 - \frac{1}{6}t^2 + \frac{1}{120}t^4$$

which is the Taylor series expansion of $\dfrac{\sin(t)}{t}$, the exact solution.
If $m = 0$, (3.1) reduces to

$$u(t) + \int_0^t (t-s)\frac{1}{s}u'(s)\,ds = 1 - \int_0^t (t-s)\,ds \tag{4.3}$$

Using $N = 4$, $J = 3$ for illustration, $I_0(t) = \begin{bmatrix} 1 & \eta_0(t) & t^2 & t^3 & t^4 \end{bmatrix}$, $\eta_0(t) =$

$\left[\frac{t^4}{24}-\frac{t^2}{2}+1\right]$. $2I_2(t)=\left[\begin{array}{ccccc} 0 & \theta_2(t) & t^2 & \frac{t^3}{2} & \frac{t^4}{3}\end{array}\right]$, $\theta_2(t)=\dfrac{t^6}{3600}-\dfrac{t^4}{72}+\dfrac{t^2}{2}$.
$g(t)=1-\dfrac{t^2}{2}$.

$\mathbf{A}=\left[\begin{array}{ccccc} 1 & 0 & \frac{-1}{6} & 0 & 0\end{array}\right]^T$. Therefore $u(4,3)=1-\dfrac{t^2}{2}$, which is the exact solution.

Example 4.3. *[7, 11, 22] Consider*

$$u''(t)+\frac{2}{t}u'(t)-2\left(2t^2+3\right)u(t)=0 \tag{4.4}$$

with $u(0)=1$, $u'(0)=0$, *the solution* $u(t)=e^{t^2}$. *On comparing with* (1.1), $h=2$. $p(t)=-2\left(2t^2+3\right)$. *The integral form is*

$$u(t)+2\int_0^t(t-s)\frac{1}{s}u'(s)\,ds-4\int_0^t(t-s)\,s^2u(s)\,ds-6\int_0^t(t-s)\,u(s)\,ds=1$$

Following the same steps in previous Examples above, we obtain the following results, we equate the coefficient of $t<e^{-0.7}$ *to zero and truncate the series at power of 10. Using* $u(10;5)$ *for illustration, Let* ${}^1I_1(t)=\int_0^t(t-s)\,s^2u(s)\,ds$, ${}^2I_1(t)=\int_0^t(t-s)\,u(s)\,ds$

$I_0(t)=\left[\begin{array}{ccccccccc} 1 & \eta_0 & t^2 & t^3 & t^4 & t^5 & t^6 & t^7 & t^8\end{array}\right]$, $\eta_0=\dfrac{t^8}{40320}-\dfrac{t^6}{720}+\dfrac{t^4}{24}-\dfrac{t^2}{2}+1$,
$2I_2=\left[\begin{array}{ccccccccc} 0 & \theta_2(t) & t^2 & 0.5t^3 & 0.333t^4 & 0.25t^5 & 0.2t^6 & 0.167t^7 & 0.148t^8\end{array}\right]$,
$\theta_2(t)=3.54e^{-6t^8}-2.78e^{-4t^6}+0.0139t^4-0.5t^2$,
$-4{}^1I_1(t)=$
$\left[\begin{array}{ccccccccc} 0.83t^5 & {}^1\eta_1 & 0.83^6 & 0.02t^7 & 0.02t^8 & 0.012t^9 & 0.01t^{10} & 0.001t^{11} & 0.001t^{12}\end{array}\right]$,
${}^1\eta_1(t)=-1.54e^{-5t^{10}}+7.44e^{-4t^8}-0.0167t^6+0.0833t^4$,
$-6{}^2I_1(t)=$
$\left[\begin{array}{ccccccccc} 0.5t^2 & {}^2\eta_1 & 0.83^4 & 0.05t^5 & 0.033t^6 & 0.024t^7 & 0.02t^8 & 0.0014t^9 & 0.011t^{10}\end{array}\right]$,
${}^2\eta_1(t)=-2.48e^{-5t^8}+0.00139t^6-0.0417t^4+0.5t^2$
$\mathbf{A}=$
$\left[\begin{array}{ccccccccc} 2.49e^5 & -2.49e^5 & -1.25e^5 & 1.58e^{-4} & 1.04e^{-4} & 0.007 & -346 & 0.03 & 6.2\end{array}\right]^T$.

Substituting back into the approximate solution gives

$$\begin{aligned} u\,(8;5) &= 0.0255t^8 + 0.0255t^7 + 0.148t^6 + 0.0073t^5 + 0.498t^4 + 0.0001t^3 \\ &\quad +0.999995t^2 + 1.0 \end{aligned}$$

In the results below, we equate the coefficient of $t < e^{-0.7}$ to zero and truncate the series at power of 10

$$\begin{aligned} u\,(15;8) &= 0.0083561578t^{10} - 5.4439212e^{-6t^9} + 0.041667245t^8 \\ &\quad +0.16666662t^6 + 0.5t^4 + t^2 + 1.0 \end{aligned}$$

$$u\,(20;8) = 0.0083335118t^{10} + 0.041666675t^8 + 0.16666667t^6 + 0.5t^4 + t^2 + 1.0$$

$$u\,(25;8) = 0.0083333334t^{10} + 0.041666667t^8 + 0.16666667t^6 + 0.5t^4 + t^2 + 1.0$$

The results obtained show that N is increasing, the accuracy is improving hence it shows that the method converges.

Example 4.4. *[15] Consider*

$$u''\,(t) + \frac{2}{t}u'\,(t) + u\,(t) = 6 + 12t + t^2 + t^3 \tag{4.5}$$

with $u\,(0) = 0$, $u'\,(0) = 0$, *the solution* $u\,(t) = t^2 + t^3$. *Taking* $N = 3$ *and* $J = 2$, *then,* $2I_2\,(t) = \begin{bmatrix} 0 & \theta_2\,(t) & t^2 & \frac{t^3}{2} \end{bmatrix}$, $\theta_2\,(t) = \dfrac{t^4}{72} + \dfrac{t^2}{2}$,

$$2\mathbf{I}_2\,(t_j) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{323}{5832} & \frac{1}{9} & \frac{1}{54} \\ 0 & \frac{160}{729} & \frac{4}{9} & \frac{4}{27} \\ 0 & \frac{35}{72} & 1 & \frac{1}{2} \end{bmatrix}.$$

Taking $r = 0$ in (3.6), $I_1\,(t) = \begin{bmatrix} \frac{t^2}{2} & \theta_1\,(t) & \frac{t^4}{12} & \frac{t^5}{20} \end{bmatrix}$, $\theta_1\,(t) = \dfrac{-t^4}{24} + \dfrac{t^2}{2}$,
$\mathbf{I}_1\,(t_j) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{18} & \frac{107}{1944} & \frac{1}{972} & \frac{1}{4860} \\ \frac{2}{9} & \frac{52}{243} & \frac{4}{243} & \frac{8}{1215} \\ \frac{1}{2} & \frac{11}{24} & \frac{1}{12} & \frac{1}{20} \end{bmatrix}$, $I_0\,(t) = \begin{bmatrix} 1 & \eta_0\,(t) & t^2 & t^3 \end{bmatrix}$, $\eta_0\,(t) =$
$1 - \dfrac{t^2}{2}$, $\mathbf{I}_0\,(t_j) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & \frac{17}{18} & \frac{1}{9} & \frac{1}{27} \\ 1 & \frac{7}{9} & \frac{4}{9} & \frac{8}{27} \\ 1 & \frac{1}{2} & 1 & 1 \end{bmatrix}$, $g\,(t) = \dfrac{t^2}{60}\left(3t^3 + 5t^2 + 120t + 180\right)$,

$\mathbf{g}\left(t_j\right) = \left[\begin{array}{cccc} 0 & \frac{331}{810} & \frac{2368}{1215} & \frac{77}{15} \end{array}\right]^T$. Therefore $\mathbf{w}\left(t_j\right) = \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ \frac{19}{18} & \frac{6475}{5832} & \frac{325}{972} & \frac{361}{4860} \\ \frac{11}{9} & \frac{1043}{729} & \frac{328}{243} & \frac{728}{1215} \\ \frac{3}{2} & \frac{139}{72} & \frac{37}{12} & \frac{41}{1215} \end{array}\right]$,

Hence $\mathbf{A} = \left[\begin{array}{cccc} 0 & 0 & 1 & 1 \end{array}\right]^T$. Substituting back into the approximate solution gives the exact solution.

Example 4.5. *[6] Consider*

$$u''\left(t\right) + \frac{2}{t}u'\left(t\right) + \left(3 - t^2\right)u\left(t\right) = 0$$

subject to the initial conditions $u\left(0\right) = 1$, $u'\left(0\right) = 0$. *The exact solution* $u\left(t\right) = e^{-t^2}$. *Following the steps in Examples* $1 - 3$, *we obtain the following results, we equate the coefficient of* $t < e^{-0.7}$ *to zero and truncate the series at power of 10*

$$\begin{aligned} &u\left(10;4\right) \\ &= -1.6030642e^{-4t^{10}} - 8.7365408e^{-5t^9} + 5.347587e^{-4t^8} - 5.6801769e^{-5t^7} \\ &- 0.020811163t^6 - 5.3737829e^{-6t^5} + 0.12500076t^4 - 0.5t^2 + 1.0 \end{aligned}$$

$$\begin{aligned} &u\left(15;6\right) \\ &= -2.6123777e^{-4t^{10}} + 0.0026040537t^8 - 0.020833338t^6 + 0.125t^4 - 0.5t^2 + 1.0 \end{aligned}$$

$$\begin{aligned} &u\left(20;6\right) \\ &= -2.6041701e^{-4t^{10}} + 0.0026041667t^8 - 0.020833333t^6 + 0.125t^4 - 0.5t^2 + 1.0 \end{aligned}$$

$$\begin{aligned} &u\left(25;6\right) \\ &= -2.6041667e^{-4t^{10}} + 0.0026041667t^8 - 0.020833333t^6 + 0.125t^4 - 0.5t^2 + 1.0 \end{aligned}$$

The results obtained is the Taylor series expansion of the approximate solution

5 Conclusion

We have discussed the development and implementation of method of approximating the solution of linear singular second order initial value problems. The approach in this paper is simple to develop and implement without loosing accuracy. The implementation is done with the aid of a MATLAB code which makes the implementation simple, easier and flexible. The limits are verified using scientific workplace software. The approximate solution effectively handle the singular points, hence the method works with all the method of choosing collocation points. Numerical results confirmed the efficiency of the new method in terms of accuracy and simplicity.

Conflict of Interest
The authors declare that there are no conflict of interest regarding the publication of this paper.

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