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Numerical method for the solution of volterra-fredholm integro-differential equations

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Abstract

In the present study, we investigate the combination of Taylor series and Block pulse functions solutions of higher order linear integro-differential Volterra-Fredholm equations (IDVFE) by using a new method. This method transforms IDVFE into the matrix equations which correspond to a system of linear algebraic equations. Some numerical results are also given to illustrate the efficiency of the method.

1 Introduction

Integro-differential equations (IDE) have many applications in different fields of study such as biological models, industrial mathematics, control theory of financial mathematics, economics, electrostatics, fluid dynamics, heat and mass transfer, oscillation theory, queuing theory.

It is usually difficult to solve IDE analytically, therefore it is better to find an efficient approximation scheme for solving such equations. There are several numerical methods for integro-differential equations such as El-gendi's, Wolfe's, Galerkin methods [1], Euler–Chebyshev [2], Runge–Kutta [3] methods, rationalized Haar functions[4] method, Galerkin methods with hybrid functions [5], Parabolic Basis

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Functions method [6], Lagrange polynomial method [7] and the Taylor collocation method [8].

A Chebyshev collocation method, which was given for the solution of the linear integro-differential equations, was developed for the system of Fredholm–Volterra IDE [9].

Taylor polynomial method was recently developed for the following single Volterra-Fredholm integral equation and integro-differential equations in real application. Yalçinbaş and Sezer [10] employed the Taylor collocation method to solve second-order linear differential equations. Sezer et al. [11, 12] also used this method in their work on linear integro-differential equations and high-order linear Fredholm–Volterra integro-differential equations, Furthermore, there is not any available research on the solution methods of the higher-order Fredholm–Volterra IDE using the hybrid of Taylor series and Block pulse functions method which will be our focus in Section 2. In this study, we have presented a numerical framework for solving the integro-differential equations by modifying diffusion the known method for integral equation. As a result, we observe that the approximation solution obtained by the present method has a good agreement with the exact solution so it provides a good approximation when compared to other methods. A considerable advantage of the method is that it allows us to make use of the computer because this hybrid of Taylor series and Block pulse functions method transform the problem into the matrix equation which is a linear algebraic system. Therefore, hybrid of Taylor series and Block pulse functions coefficients of the solution are found very easily by using the computer programs.

In this paper, we will consider linear IDEs of Fredholm–Volterra type in the form

$$\sum_{n=0}^{2} P_n(x)\, y^{(n)}(x) = g(x) + \int_0^x K(x,s)\, y(s)\, ds + \int_0^1 F(x,s)\, y(s)\, ds, \quad 0 < x < 1, \tag{1.1}$$

under the initial conditions. Consider the Taylor polynomials $T_m(t) = t^m; m = 0, 1, 2, 3, \ldots,$ on the interval $[0, 1]$. A set of block pulse functions $\varphi_i(t); i = 1, 2, \ldots, m$ and the set of functions $h_{ij}(t); j = 0, \ldots, (M-1), i = 1, 2, \ldots, N$ that produces by hybrid of Taylor series and Block pulse functions on the interval $[0, 1]$ are defined as follows respectively:

$$\varphi_i(t) = \begin{cases} 1 & ; \ \frac{i-1}{m} \le t < \frac{i}{m} \\ 0 & ; \ otherwise \end{cases}, \tag{1.2}$$

$$h_{ij}\left(t\right)=\begin{cases}T_j(Nt-i+1) & ; \quad \frac{i-1}{N}\le t<\frac{i}{N}\\ 0 & ; \quad otherwise\end{cases}, \tag{1.3}$$

a function $y\left(x\right)\in L^2([0,1])$ can be approximated as

$$y\left(x\right)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}y_{ij}h_{ij}\left(x\right), \tag{1.4}$$

where

$$y_{ij}=\frac{1}{N^j j!}\left(\frac{d^j y(x)}{dx^j}\right)\bigg|_{x=\frac{i-1}{N}},$$

if $y\left(x\right)$ is in Eq. (1.4) is truncated, then Eq. (1.4) can be written as

$$y\left(x\right)\approx\sum_{i=1}^{N}\sum_{j=0}^{M-1}y_{ij}h_{ij}\left(x\right)=H(x)Y, \tag{1.5}$$

where

$$Y=[y_{10},\ldots,y_{1(M-1)},y_{20},\ldots,y_{2(M-1)},\ldots,y_{N0},\ldots,y_{N(M-1)}]^T, \tag{1.6}$$

and

$$H\left(t\right)=\left[h_{10},\ldots,h_{1(M-1)},h_{20},\ldots,h_{2(M-1)},\ldots,h_{N0},\ldots,h_{N(M-1)}\right]^{\square}. \tag{1.7}$$

We also approximate the function $K(x,s)\in L^2([a,b]\times[a,b])$ as follows

$$K\left(x,s\right)\approx H\left(x\right)KH\left(s\right)^T, \tag{1.8}$$

where K is an $NM\times NM$ matrix that

$$K_{ij}=\frac{1}{N^{r+m}r!m!}\left(\frac{d^{i+j}K(x,s)}{dx^i ds^j}\right)\bigg|_{(x,s)=(\frac{i}{N},\frac{i}{N})}, \tag{1.9}$$

such that

$$i,j=0,1,2,\ldots,NM-1, r=i-\left[\frac{i}{N}\right]N, m=j-\left[\frac{j}{N}\right]N.$$

The integration of the $H(t)$ defined in Eq. (1.7) can be approximated by

$$\int_0^t H(\tau)\,d\tau = PH^T(t) = H(t)P^T, \tag{1.10}$$

where P is an $NM \times NM$ operational matrix for integration and is given by

$$P = \begin{bmatrix} E & H & \dots & H \\ 0 & E & \cdots & H \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E \end{bmatrix}_{NM \times NM} \tag{1.11}$$

with

$$H = \frac{1}{N} \begin{bmatrix} 1 & 0 & \dots & 0 \\ \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{M} & 0 & \cdots & 0 \end{bmatrix}_{M \times M} \tag{1.12}$$

and E is the operational matrix of integration for Taylor polynomials on the interval $\left[\frac{i-1}{N}, \frac{i}{N}\right]$ which is given in [13, 14] by

$$E = \begin{bmatrix} 010 & \dots & 0 \\ 00\frac{1}{2} & \cdots & \vdots \\ \vdots & \ddots & \frac{1}{M-1} \\ 000 & \cdots & 0 \end{bmatrix}_{M \times M} . \tag{1.13}$$

2 The method of the solution

In this section, we will modify the hybrid of Taylor series and Block pulse functions for single Volterra-fredholm integral equation to the current system of higher order of Volterra-Fredholm integro-differential equations.

The functions $y^{(2)}(x)$ used in (1.1) can be written in the matrix form

$$y^{(2)}(x) \approx H(x)\,Y, \tag{2.1}$$

Integrating Eq. (14) from 0 to *x* and using Eq. (11) we obtain

$$\acute{y}(x) \approx H(x)\,P^T Y + \acute{y}(0)\,, \tag{2.2}$$

$$y\left(x\right)\approx H\left(x\right)\left(P^{T}\right)^{2}Y+\acute{y}\left(0\right)x+y\left(0\right), \tag{2.3}$$

where *P* is the operational matrix of integration given in Eq. (1.11).

The functions $\int_0^x K\left(x,s\right)y\left(s\right)ds$ and $\int_0^1 F\left(x,s\right)y\left(s\right)ds$ can be written in matrix form

$$\int_0^x K\left(x,s\right)y\left(s\right)ds=K_1\left(x\right)Y+K_2\left(x\right), \tag{2.4}$$

$$\int_0^1 F\left(x,s\right)y\left(s\right)ds=F_1\left(x\right)Y+F_2\left(x\right), \tag{2.5}$$

where

$$\begin{aligned}K_1\left(x\right)&=\left[\int_0^x K\left(x,s\right)H(s)\left(P^{T}\right)^{2}ds\right]_{1\times NM},\\ K_2\left(x\right)&=\int_0^x K\left(x,s\right)\left(\acute{y}\left(0\right)s+y(0)\right)ds,\end{aligned} \tag{2.6}$$

$$\begin{aligned}F_1\left(x\right)&=\left[\int_0^1 F\left(x,s\right)H(s)\left(P^{T}\right)^{2}ds\right]_{1\times NM},\\ F_2\left(x\right)&=\int_0^1 F\left(x,s\right)\left(\acute{y}\left(0\right)s+y(0)\right)ds,\end{aligned} \tag{2.7}$$

By substituting Equations (2.1), (2.2), (2.3), (2.4) and (2.5) into Eq. (1), we obtain

$$Q_1\left(x\right)Y+Q_2\left(x\right)=g(x)+K_1\left(x\right)Y+K_2\left(x\right)+F_1\left(x\right)Y+F_2(x), \tag{2.8}$$

where

$$Q_1\left(x\right)=P_2\left(x\right)H\left(x\right)+P_1\left(x\right)H\left(x\right)P^{T}+P_0\left(x\right)H\left(x\right)\left(P^{T}\right)^{2}, \tag{2.9}$$

$$Q_2\left(x\right)=P_1\left(x\right)\acute{y}\left(0\right)+P_0\left(x\right)\left(\acute{y}\left(0\right)x+y\left(0\right)\right). \tag{2.10}$$

Then the system (2.8) can be rewritten as

$$Q(x)Y=G\left(x\right), \tag{2.11}$$

where

$$Q\left(x\right)=Q_1\left(x\right)-K_1\left(x\right)-F_1\left(x\right),G(x)=g(x)+K_2(x)+F_2(x)-Q_2(x). \tag{2.12}$$

2.1 Application of the hybrid of Taylor series and Block pulse functions method

To compute the hybrid of Taylor series and Block pulse functions coefficients we use the collocation points defined by $x_r = rh, r = 1, \ldots, NM$ where $h = \frac{2}{NM}$ and the values x_r are spread out over the interval [0, 1]. Substituting the hybrid of Taylor series and Block pulse functions points into (2.11), we obtain the following matrix form

$$QY = G, \tag{2.13}$$

where

$$Q = \begin{bmatrix} (Q_1 - K_1 - F_1)(x_0) \\ (Q_1 - K_1 - F_1)(x_1) \\ \vdots \\ (Q_1 - K_1 - F_1)(x_{NM}) \end{bmatrix}, \tag{2.14}$$

$$G = \begin{bmatrix} (g + K_2 + F_2 - Q_2)(x_0) \\ (g + K_2 + F_2 - Q_2)(x_1) \\ \vdots \\ (g + K_2 + F_2 - Q_2)(x_{NM}) \end{bmatrix}. \tag{2.15}$$

Eq. (2.13) is the fundamental matrix equation for the Volterra-Fredholm integro-differential equations.

Which yield a system of linear algebraic equations of unknown hybrid of Taylor series and Block pulse functions coefficients.

If the matrix Q is nonsingular then we can write $Y = Q^{-1}G$ and the hybrid of Taylor series and Block pulse functions coefficients without initial conditions are determined. Thus the hybrid of Taylor series and Block pulse functions are uniquely determined.

An interesting feature of this method is that when an integral equation has linearly independent polynomial solution of degree n or less than n, the method can be used for finding the analytical solution. The suggested expansion method is closer to the exact solution.

Comparison of the results obtained from this method with another methods indicate that the suggested method has simple algorithm and this method can be applied efficiently to a variety of similar problems.

3 Numerical examples

In this section, we state the numerical results for Volterra-Fredholm integro-differential equations.

Example 3.1. *Let us now consider the linear second order FVIDE for* $0 \leq x, t \leq 1$ *given by*

$$y^{(2)}(x) + xy^{(1)}(x) - xy(x) = e^{x} - \sin(x) + \frac{1}{2} x cos(x)$$
$$+ \int_0^1 \sin(x) e^{-s} y(s) \, ds - \frac{1}{2} \int_0^x (\cos [2061?](x) e^{-s} y(s)) ds,$$

with the initial conditions $y(0) = 1$ *and* $\acute{y}(0) = 1$ *which is the exact solution* $y(x) = e^{x}$.

Here, $g(x) = e^{x} - \sin(x) + \frac{1}{2} x cos(x)$, $K(x,s) = \frac{-1}{2} \cos [2061?](x) e^{-s}$, $F(x,s) = \sin(x) e^{-s}$, $P_2(x) = 1$, $P_1(x) = x$ *and* $P_0(x) = -x$.

Let us approximate the solution by hybrid of Taylor series and Block pulse functions.

Table 1 shows the approximate solution by present method for N =6, M= 3 and N =8, M= 5.

Table 1. Numerical results of the absolute error functions

X	CAS wavelet method [15]	Differential transformation [16]	HPM [17]	Present method (N=6, M=3)	Present method (N=8, M=5)
0.1	$1.34917637e^{-03}$	$1.00118319e^{-02}$	$2.314814815e^{-06}$	$3.2481e^{-04}$	$5.7374e^{-08}$
0.2	$1.15960044e^{-03}$	$2.78651355e^{-02}$	$9.259259259e^{-06}$	$1.2172e^{-04}$	$2.2371e^{-08}$
0.4	$5.93105645e^{-02}$	$7.55356316e^{-02}$	$3.703703704e^{-05}$	$4.7262e^{-04}$	$6.7251e^{-08}$
0.6	$4.39287720e^{-02}$	$1.09551714e^{-01}$	$8.333333333e^{-05}$	$3.2475e^{-04}$	$2.3664e^{-08}$
0.8	$1.34514117e^{-02}$	$6.94512700e^{-02}$	$1.481481481e^{-04}$	$4.3150e^{-04}$	$4.3141e^{-08}$
0.9	$1.32045209e^{-02}$	$1.00034260e^{-02}$	$1.875000000e^{-04}$	$2.3401e^{-04}$	$2.6121e^{-07}$

In addition, the numerical results of the absolute error functions obtained by the present method for N =6, M= 3 and N =8, M= 5, the CAS wavelet method [15], the differential transformation [16] and the HPM [17] are compared in Table 1. It is seen from Table 1 that the results obtained by the present method is better than that obtained by the other methods.

Example 3.2. *Consider the linear Fredholm integro-differential equation given by*

$$y^{(1)}(x) = xe^{x} + e^{x} - x + \int_{0}^{1} xy(t)\,dt, 0 \leq x, t \leq 1$$

with the initial condition $y(0) = 1$ *and the exact solution* $y(x) = xe^{x}$. *Here,* $g(x) = xe^{x} + e^{x} - x$, $F(x,t) = x$ *and* $P_1(x) = 1$.

Table 2. Numerical results of the absolute error functions

X	Exact solution	CAS wavelet method [15]	Error of CAS wavelet method	Error of Present method (N=8, M=5)
0.1	0.11051709	0.10050526	1.00118319e^{-02}	3.2118e^{-07}
0.2	0.24428055	0.21641542	2.78651355e^{-02}	2.0346e^{-07}
0.4	0.59672988	0.52119425	7.55356316e^{-02}	4.1376e^{-07}
0.6	1.09327128	0.98371957	1.09551714e^{-02}	1.4251e^{-07}
0.8	1.78043274	1.71098147	0.94512700e^{-02}	1.0251e^{-06}
0.9	2.21364280	2.22364623	1.00034260e^{-02}	3.4652e^{-06}

Example 3.3. *Consider the linear Voltra integro-differential equation given by*

$$y^{(1)}(x) = 1 - \int_{0}^{x} y(t)\,dt, 0 \leq x, t \leq 1$$

with the initial condition $y(0) = 0$ *and the exact solution* $y(x) = \sin(x)$. *Here,* $g(x) = 1$, $F(x,t) = -1$ *and* $P_1(x) = 1$.

Table 3. Numerical results of the absolute error functions

x	Exact solution	Taylor method [18] (N=5)	Error of Taylor method	Error of Present method (N=8, M=5)
0.1	0.0998334166	0.09983341667	$0.2e^{-10}$	$2.3938e^{-07}$
0.2	0.1986693308	0.1986693333	$0.25e^{-8}$	$4.0140e^{-07}$
0.4	0.3894183423	0.3894186667	$0.3244e^{-6}$	$2.0474e^{-07}$
0.6	0.5646424734	0.5646480000	$0.55266e^{-5}$	$3.6576e^{-07}$
0.8	0.7173560909	0.7173973333	$0.412424e^{-4}$	$1.7172e^{-07}$
0.9	0.7833269096	0.7834207500	$0.938404e^{-4}$	$1.1776e^{-06}$

4 CONCLUSIONS

High order integro-differential equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. For this purpose, the presented method can be proposed.

The method presented in this study is a method for computing the coefficients in the hybrid of Taylor series and Block pulse functions of the solution of a linear integro-differential equation. To obtain the best approximating solution of the equation, we take more terms from the hybrid of Taylor series and Block pulse functions of functions; that is, the truncation limit N must be chosen to be large enough.

The method can also be extended to the partial integro-differential equations and to the system of ordinary differential equations with variable coefficients.

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