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POLOIDS AND MATRICES

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Abstract

This study is about the poloids that are obtained by the formation of a new algebraic structure obtained by adding the condition (G4) obtained from the solution of the equation $XA = B$ and $B|A$ to the definition of monoid. The (G4) property is based on the factorial property of a noncommutative matrix. The (G4) property is based on the factorial property of a noncommutative matrix. Divisibility in matrices contributes to the existence of common factors of a matrix. This necessitates the distinguishing feature in data in theoretical and applied computer sciences. For example, it paves the way for detecting the truth of lying in the syntax of the person who is lying.

1 Introduction

Here is a brief history of the monoid. The name "monoid" was first used in mathematics by Arthur Cayley for a surface of order n which has a multiple point of order $n-1$.

In the context of semigroups the name is due to Bourbaki.

It is also worth commenting on the related term monoid, meaning an associative magma with identity. This term is a little more recent than semigroup, and seems to originate with Bourbaki. Before this, Birkhoff (1934) was using the term groupoid

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for an associative magma with identity. More precisely Bourbaki (1942, p. 7): A set endowed with the structure determined by an associative law every where defined takes the name of monoid. Perhaps this was motivated by Eilenberg & Mac Lanes upcoming A monoid is a category with one object? (They started categories around 1942.) [2, 3].

The monoid, which briefly forms the algebraic structures of mathematics defined by a binary operation, is the basis for the study of monoids, automata theory (Krohn-Rhodes theory) and formal language theory (star height problem) in theoretical computer science.
The definition of poloid, defined by this binary operation, which includes a monoid, was discovered during work on factoring a matrix. It is defined by us, considering that it will make a wider contribution to theoretical computer science and formal language theory. The (G4) condition added to a monoid definition preserves the algebraic structure being applied in computer science, and also offers new unobservable paths and alternative options.

Let us start with the row co-divisor definition that I gave in the study in 2022.

Here F is a field and $M_n(F) = \left\{ [a_{ij}]_n \middle| a_{ij} \in F, n \in \mathbb{Z}^+ \right\}$ is the set of regular matrices. The transpose of $A \in M_n(F)$ is denoted by A^T.

Let A and B be two regular square matrices of order n. The determinant of the new matrix obtained by writing the i^{th} row of the matrix A on the j^{th} row of the matrix B is called the *co-divisor by row* of the matrix A by the row on the matrix B. It denoted by $\underset{ij}{AB}$. Their number is n^2. The matrix co-divisor by row is $\left[\left(\underset{ij}{AB}\right)_{ij}\right]$ [9].

Example 1.1. *Let* $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix}$ *be regular matrices. Matrix of co-divisors by row of matrix* A *on matrix* B *is* $\left[\left(\underset{ij}{AB}\right)_{ij}\right]$.

$$\underset{11}{AB} = \begin{vmatrix} 1 & 3 \\ 4 & 7 \end{vmatrix} = -5, \underset{12}{AB} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5. \underset{21}{AB} = \begin{vmatrix} 2 & 5 \\ 4 & 7 \end{vmatrix} = -6, \underset{22}{AB} = \begin{vmatrix} 2 & 1 \\ 2 & 5 \end{vmatrix} = 8.$$

$$\left[\left(\underset{ij}{BA} \right)_{ij} \right] = \begin{bmatrix} -5 & 5 \\ -6 & 8 \end{bmatrix}$$

Likewise, the matrix of rows co-dividing matrix B over matrix A is a matrix $\left[\left(\underset{ij}{BA} \right)_{ij} \right]$.

$$\underset{11}{BA} = \begin{vmatrix} 2 & 1 \\ 2 & 5 \end{vmatrix} = 8, \underset{12}{BA} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5. \underset{21}{BA} = \begin{vmatrix} 4 & 7 \\ 2 & 5 \end{vmatrix} = 6, \ \underset{22}{BA} \begin{vmatrix} 1 & 3 \\ 4 & 7 \end{vmatrix} = -5.$$

$$\left[\left(\underset{ij}{AB} \right)_{IJ} \right] = \begin{bmatrix} 8 & -5 \\ 6 & -5 \end{bmatrix}.$$

For the two matrices satisfying the above conditions, the matrix division is also given by $\frac{A}{B} := \frac{1}{|B|} \left[\left(\frac{A_i}{B_j} \right)_{ji} \right]$ and at the same time, the solution of the equation $AX = B$ is $X = \frac{B}{A}[4, 7, 6, 5, 8]$.

Volodymyr P. Shchedryk gave the following proved theorem in [12]. It is been determined that this theorem has to do with column division. The proof is given in my study called " Different Approaches on the Matrix Division and Generalization of Cramers Rule" in 2017 [5].

Lemma 1.1. *Let $A, B \in M_n(F)$. If $B|A$, then it is $A|B$.*

Proof. For all $A, B \in M_n(F)$, If $B|A$ then,

$$B|A \Leftrightarrow \exists T \in M_n(F) : A = BT$$
$$A = BT \Leftrightarrow A \left(\frac{I_n}{T} \right) = B \Leftrightarrow A|B.$$

□

Theorem 1.1. *Let R be a commutative elementary divisor domain. If $BX = A$ is a solvable matrix equation over R, where $A, B, X \in M_n(F)$ then a left g.c.d. and* *a left l.c.m. of solutions of this equation are also solutions of $BX = A$ [10].*

The solution of the equation $BX = A$ is $X = [x_{ij}] = \left[\frac{\left(\frac{A_i}{B_j}\right)_{ji}}{|B|}\right]$ in terms of column co-divisors and $X = \frac{A}{B}$ according to the division operation [5,8,9].

Lemma 1.2. *Let $A \in M_n(F)$.Then,*

$$\left(\frac{I_n}{A}\right)^T = \frac{I_n}{A^T}.$$

Proof. Let a regular matrix $A = [a_{ij}]_n$ be given.

$$\frac{I_n}{A^T}.A^T = I_n \wedge A^T.\frac{I_n}{A^T} = I_n$$

$$\frac{I_n}{A^T} = \left(A^T\right)^{-1} = \left(A^{-1}\right)^T = \left(\frac{I_n}{A}\right)^T$$

$$\frac{I_n}{A^T} = \left(\frac{I}{A}\right)^T.$$

The following lemma is given which simply explains the relationship between the row co-divisors matrix and the transpose. □

Lemma 1.3. *Let $A, B \in M_n(F)$. Then,*

$$\frac{1}{|A|}\left[\left(\underset{ij}{BA}\right)_{ij}\right] = \left(\frac{B^T}{A^T}\right)^T.$$

Proof. For all $A, B \in M_n(F)$ then $\underset{ij}{BA} = \frac{B^T}{A^T}ij$. Because, the row co-divisors of matrix B on matrix A are the same as the column co-divisors of matrix B^T on matrix A^T.

$$\left[\left(\underset{ij}{BA}\right)\right] = \left[\left(\frac{B^T}{A^T}ij\right)_{ij}\right]$$

$$\frac{1}{|A|}\left[\left(\underset{ij}{BA}\right)_{ij}\right] = \frac{1}{|A^T|}\left[\left(\frac{B^T}{A^T}ij\right)\right]^T = \left(\frac{B^T}{A^T}\right)^T.$$

□

Proposition 1.1. *Let $A, B \in M_n(F)$. Then, the solution of the linear matrix equation $XA = B$*

$$X = \left(\frac{B^T}{A^T}\right)^T .$$

Proof. The solution of the equation $AX = B$ is $X = \frac{B}{A}$, for all $A, B \in M_n(F)$. Then

$$XA = B \Leftrightarrow (XA)^T = B^T \Leftrightarrow A^T X^T = B^T$$

$$X^T = \frac{1}{|A^T|}\left[\left(B^T_{\ ij} A^T\right) ij\right] \Rightarrow X = \frac{1}{|A_T|}\left[\left(B^T_{\ ij} A^T\right)_{ji}\right]^T$$

$$X = \frac{1}{|A^T|}\left[\left(B^T_{\ ij} A^T\right)_{ij}\right] = \left(\frac{B^T}{A^T}\right)^T .$$

□

Due to the properties as given is [8], the following Proposition regarding the solution of this equation is obtained.

Proposition 1.2. *Let $A, B \in M_n(F)$. If the factors of matrix A is BA_1 and the factors of matrix B is AB_1 then*

(i) The rational matrix $\frac{A}{B}$ is equal to matrix A_1.

(ii) The rational matrix $\frac{A}{B}$ is equal to matrix $\frac{I_n}{B_1}$.

Proof. (i) The matrix A is written in terms of B as $A = BA_1$.

$$\frac{A}{B} = \frac{BA_1}{B} = A_1.$$

(ii) The matrix B is written in terms of A as $B = AB_1$.

$$\frac{A}{B} = \frac{A}{AB_1} = \frac{I_n}{B_1}.$$

□

Theorem 1.2. *Let $A, B, X \in M_n(F)$ and X unknowns matrix. Then, in the solution of the equation $AX = B$, there are regular matrices $A = B_2A_3$, $B =$*

B_2B_3, such as B_2, A_3 and B_3, and the rational matrix $\frac{B_3}{A_3}$ is the solution of the equation $AX = B$. This solution is equal to the rational matrix $\frac{B}{A}$.

Proof. Since the solution of $Ax = B$ is the rational matrix $\frac{B}{A}$, where any factor of matrix B is matrix B_2

$$B = B_2B_3,$$

Likewise, matrix A in terms of this B_2 matrix multiplier.
It can be written as

$$A = B_2A_3$$

Therefore

$$X = \frac{B}{A} = \frac{B_3}{A_3}.$$

□

Example 1.2. *Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix}$ be two matrices in $M_2(F)$, the solution of the equation $AX = B$ is $X = \frac{B}{A}$. If $B_2 = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ is selected*

$$B = B_2 \underbrace{\begin{bmatrix} -\frac{8}{3} & -\frac{19}{3} \\ \frac{10}{3} & \frac{20}{3} \end{bmatrix}}_{B_3}$$

Likewise,

$$A = B_2 \underbrace{\begin{bmatrix} -\frac{4}{3} & -3 \\ \frac{5}{3} & 4 \end{bmatrix}}_{A_3}$$

Then we obtain,

$$X = \frac{B}{A} = \frac{B_2B_3}{B_2A_3} = I_3\frac{B_3}{A_3} = \frac{B_3}{A_3} = \begin{bmatrix} 2 & 16 \\ 0 & -5 \end{bmatrix}$$

2 Matrix Poloids

Let's start this section with the following definition.

Definition 2.1. *A group is a set G equipped with a binary operation $\cdot : G \times G \to G$ that associates an element $a.b \in G$ to every pair of elements $a, b \in G$, and having the following properties: is associative, has an identity element $e \in G$, and every element in G is invertible (w.r.t.). More explicitly, this means that the following equations hold for all $a, b, c \in G$:*

(G1) $a.(b.c) = (a.b).c$. *(associativity);*

(G2) $a.e = e.a = a$ *(identity);*

(G3) For every $a \in G$, there is some $a^{-1} \in G$ such that $a.a^{-1} = a^{-1}.a = e$ (inverse)[11].

A set M together with an operation $: \cdot M \times M \to M$ and an element e satisfying only Conditions (G1) and (G2) is called a monoid [1].

Noticed that if the conditions for G1 and G2 are met on the multiplication operation in matrices. So let's briefly examine whether it is a monoid or not. The set of $M_n(F)$-square matrices satisfies the conditions (G1) and (G2), However, the following example has $A = BA_1$, and $A = A_1C$ whereas $B \neq C$.

Example 2.1. *We have matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ written as follows:*

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \frac{4}{3} & 2 \\ \frac{1}{3} & 0 \end{bmatrix}}_{A_1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A,$$

And it is also,

$$\underbrace{\begin{bmatrix} \frac{4}{3} & 2 \\ \frac{1}{3} & 0 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} 9 & 12 \\ -\frac{11}{2} & -7 \end{bmatrix}}_{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A.$$

Here, neither $B = C$ nor $A_1 = I_n$ nor $A_1 = A$.

To further explore this expression, the following new definition is given.

Definition 2.2. *A group is a set G equipped with a binary operation $* : G \times G \to G$ that associates an element $a.b \in G$ to every pair of elements $a, b \in G$, and having* *the following properties: is associative, has an identity element $e \in G$, and every element in G is invertible (w.r.t. $*$). More explicitly, this means that the following equations hold for all $a, b, c, d, e \in G$:*

(G1) $a * (b * c) = (a * b) * c$. *(associativity)*

(G2) $a * e = e * a$. *(identity);*

(G3) For every $a \in G$, there is some $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$. (inverse)

(G4) For every $a \in G$, there some $d, f \in G$ such that $b * f = f * d = a$ with $b \neq d$. (escort).

A set M together with an operation $* : G \times G \to G$ and an element e satisfying only Conditions (G1), (G2), (G3) and (G4) is called a poloid. It is denoted by $(G, *)$.

Example 2.2. *For example, the set $M_n(\mathbb{R})$ of square matrices is poloid under multiplication. But, the set of real numbers $\mathbb{R}$ is not poloid by multiplication. Let us take the real number 2.*

$$2 = \frac{1}{3}.6 \wedge 2 = \frac{12}{2}.\frac{1}{2}$$

Here although

$$6 = \frac{12}{2}$$

The condition (G4) is not satisfied. Therefore, every poloid is also a monoid. The converse of the statement is not always true.

The set $M_n(F)$ is poloid when the multiplication operation in the matrices is considered.

Lemma 2.1. *Let $M_n(F)$ be a poloid. For all $A_1 \in M_n(F)$ then,*

(i) There are $A, C \in M_n(F)$ regular matrices such that $A_1 = \frac{A}{C}$.

(ii) There are $A, C \in M_n(F)$ regular matrices such that $A_1 = \frac{1}{|C|}\left[\left(\underset{ij}{AC}\right)_{ij}\right]$.

Proof. The proofs of (i) and (ii) are easily obtained from Lemma 2. □

Theorem 2.1. *Let $M_n(F)$ be a poloid. Then, there are matrices $A, C \in M_n(F)$ such that $A_1 = \frac{1}{|C|}\left[\begin{pmatrix} AC \\ ij \end{pmatrix}_{ij}\right] = \begin{pmatrix} \frac{A^T}{C^T} \end{pmatrix}$, for all $A_1 \in M_n(F)$.*

Proof. The proof of the theorem 5 is easily obtained from Lemma 2, Lemma 3 and (G4). □

Theorem 2.2. *Let $M_n(F)$ be a poloid. Then, there are matrices $B, C \in M_n(F)$ such that satisfying the equation $A = BAC$, for all $A \in M_n(F)$.*

Proof. For all $A \in M_n(F)$, there are $S, R, A_1 \in M_n(F)$ such that $A = SA_1 = A_1R$ from (G4)

$$C|S \Rightarrow S = CS_1, \text{ where } S_1 \in M_n(F).$$

$$A = SA_1 = CS_1A_1,$$

$$A|S_1 \Rightarrow S_1 = AS_2, \text{ where } S_2 \in M_n(F)$$

$$A = SA_1 = CAS_2A_1, B := S_2A_1$$

$$A = SA_1 = CAB.$$

$$C|A \Leftrightarrow A_1 = CC_1, \text{ where } C_1 \in M_n(F)$$

$$A = A_1R = CC_1R \wedge A|C_1 \Leftrightarrow C_1 = AC_2, \text{ where } C_2 \in M_n(F)$$

$$A = A_1R = CAC_2R, B' = C_2R.$$

□

We want to prove that $B = B'$. Assume that B is not equal to B'. Then the fact that the B matrix is different in the B' matrix contradicts the (G4) condition.

Theorem 2.3. *Let $M_n(F)$ be a poloid. Then, there are matrices $K, L \in M_n(F)$ such that satisfying the equations $A = AKC = BLA$, for all $A \in M_n(F)$.*

Proof. It is clear if A is the unit matrix and the zero matrix. Since (G4) is provided

$$A_1 = AK \Rightarrow KC = I_n \Rightarrow A_1C = AKC = A,$$

and because of Lemma 4, $A_1 = LA \Rightarrow L = \frac{1}{|A|}\left[\left(\underset{ij}{A_1 A}\right)_{ij}\right]$ [6].

$$BL = I_n \Rightarrow BA_1 = BLA = A.$$

□

Example 2.3. *We have the matrix* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *written as follows:*

$$\underbrace{\begin{bmatrix} \frac{4}{3} & 2 \\ \frac{1}{3} & 0 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} 9 & 12 \\ -\frac{11}{2} & -7 \end{bmatrix}}_{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

$$A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -\frac{7}{3} & -4 \\ \frac{11}{6} & 3 \end{bmatrix} \begin{bmatrix} \frac{4}{3} & 2 \\ \frac{1}{3} & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} -\frac{7}{3} & -4 \\ \frac{11}{6} & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} -\frac{7}{3} & -4 \\ \frac{11}{6} & 3 \end{bmatrix} \begin{bmatrix} 9 & 12 \\ -\frac{11}{2} & -7 \end{bmatrix} = I_2$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -\frac{7}{3} & -4 \\ \frac{11}{6} & 3 \end{bmatrix} \underbrace{\begin{bmatrix} 9 & 12 \\ -\frac{11}{2} & -7 \end{bmatrix}}_{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

And it is also,

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \frac{4}{3} & 2 \\ \frac{1}{3} & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

$$A_1 = L \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$L = \frac{1}{|A|}\left[\left(\underset{ij}{A_1 A}\right)_{ij}\right] = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = I_2$$

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}}_{B} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A.$$

Conclusions and Discussions

It is obvious that the concept of "poloid", which has just been defined as our knowledge, will find many application areas. The existence of an algebraic structure that manifests itself when any element is processed from the right and left is still an open problem.

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