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# Fixed Point Theorem for Reich Contraction Mapping in Convex $b$-Metric Spaces

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**Abstract**

In this paper, we prove fixed point theorems for the Reich contraction mapping using Mann iteration sequence in a convex $b$-metric space. Also, we give the weak $T$-stability of Mann iteration for this mapping in complete convex $b$-metric spaces.

## 1 Introduction and preliminaries

In last years, several generalizations of standard metric spaces have published. Ever since S. Banach [1] proved the Banach fixed point theorem in 1922, many authors have tried to generalize this conclusion. Different studies have been made using varied generalizations of the contraction mappings in Banach fixed point theory. In 1993, Czerwik [2] introduced the concept of $b$-metric space and proved fixed point theorems in this space. It was also extended to several $b$-metric spaces with different structure on it in the past years. Some fixed point results were proved in the setting of $b$-metric spaces by many authors (see, e.g., [20, 21, 22, 23, 24]).

In 1970, Takahashi [16] introduced the concept of convex structure and convex metric space and proved some fixed point theorems for nonexpansive mappings in the convex metric space. In 1988, Xie [17] discovered existence of fixed points of quasi-contraction mappings in convex metric spaces by Ishikawa iteration process.

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In 2020, Chen et al. [4] introduced convex $b$-metric space and proved some fixed point theorems. Moreover, a lot of authors generalized the metric space into many other spaces (see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]).

The aim of this paper is to give fixed point theorems for the Reich contraction mapping in convex $b$-metric space. And also, we will show the weak $T$-stability of Mann iteration in this space. Our results expand the results in [4].

The following definitions and results will be needed in the sequel.

Let $(U, \rho)$ be a metric space and let $T : U \to U$ be a mapping. The mapping $T$ is said to be Reich [19] contraction mapping if

$$\rho(Tu, Tv) \leq k\rho(u, v) + t\left[\rho(u, Tu) + \rho(v, Tv)\right] \tag{1.1}$$

for all $u, v \in U$, where $k + 2t < 1$.

**Definition 1.1.** *[2, 3] Let $U$ be a (nonempty) set and $s \geq 1$ be a given real number. A function $\rho_b : U \times U \to [0, \infty)$ is a b-metric on $U$ if for all $u, v, y \in U$, the following conditions hold:*

*($b_1$) $\rho_b(u, v) = 0$ if and only if $u = v$,*

*($b_2$) $\rho_b(u, v) = \rho_b(v, u)$,*

*($b_3$) $\rho_b(u, y) \leq s[\rho_b(u, v) + \rho_b(v, y)]$.*

Then, $(X, \rho_b)$ is called a $b$-metric space. It is clear that a $b$-metric space is a metric space for $s = 1$.

**Definition 1.2.** *[2, 3] Let $(U, \rho_b)$ be a b-metric space and $\{u_n\}$ a sequence in $U$. We say that*

*(1) $\{u_n\}$ converges to $u \in U$ if $\rho_b(u_n, u) \to 0$ as $n \to \infty$;*

*(2) $\{u_n\}$ is a Cauchy sequence if $\rho_b(u_m, u_n) \to 0$ as $m, n \to \infty$;*

*(3) $(U, \rho_b)$ is complete if every Cauchy sequence in $U$ is convergent.*

Takahashi's [16] concept of a convex structure is following:

**Definition 1.3.** *[16] Let $U \neq \varnothing$ and $I = [0, 1]$. Define the mapping $\rho_b : U \times U \to [0, \infty)$ and a continuous function $\omega : U \times U \times I \to U$. Then $\omega$ is said to be the convex structure on $U$ if the following holds:*

$$\rho_b\left(y, \omega(u, v; \lambda)\right) \leq \lambda\rho_b(y, u) + (1 - \lambda)\rho_b(y, v) \tag{1.2}$$

*for each $y \in U$ and $(u, v; \lambda) \in U \times U \times I$.*

Chen et al. [4] gave definition of the convex $b$-metric space as follows:

**Definition 1.4.** *Let the mapping $\omega : U \times U \times I \to U$ be a convex structure on $b$-metric space $(U, \rho_b)$ with constant $s \geq 1$ and $I = [0, 1]$. Then $(U, \rho_b, \omega)$ is said to be a convex $b$-metric space.*

Let $(U, \rho_b, \omega)$ be a convex $b$-metric space and $T : U \to U$ be a mapping. The generalization of the Mann iteration process to convex $b$-metric space is following:

$$u_{n+1} = \omega\left(u_n, Tu_n; \alpha_n\right), \; n \in \mathbb{N},$$

where $u_n \in U$ and $\alpha_n \in [0, 1]$.

**Lemma 1.1.** *[18] Let $\{k_n\}$, $\{l_n\}$ be non-negative sequences satisfying $k_{n+1} \leq hk_n + l_n$ for all $n \in \mathbb{N}$, $0 \leq h < 1$, $\lim_{n\to\infty} l_n = 0$. Then $\lim_{n\to\infty} k_n = 0$.*

In 2008, Qing and Rhoades [15] introduced the concept of $T$-stability of the iteration procedure in complete metric spaces in the following:

**Definition 1.5.** *[15]Let $T$ be a self-map on a complete metric space $(U, \rho_b)$. Assume that $u_{n+1} = f(T, u_n)$ is an iteration sequence, which yields a sequence $u_n$ of points from $U$. Then the sequence $u_{n+1} = f(T, u_n)$ is said to be $T$-stable if $\{u_n\}$ convergence to a fixed point $u^*$ of $T$, and if $\{v_n\}$ is a sequence in $U$ such that $\lim_{n\to\infty} \rho_b(v_{n+1}, f(T, v_n)) = 0$, then we have $\lim_{n\to\infty} v_n = u^*$.*

In 2020, Chen et al. [4] introduced the concept of the weak $T$-stability of the iteration procedure.

**Definition 1.6.** *[4] Let $T$ be a self-map on a complete metric space $(U, \rho_b)$. Assume that $u_{n+1} = f(T, u_n)$ is an iteration sequence, which yields a sequence $u_n$ of points from $U$. Then the iteration procedure $u_{n+1} = f(T, u_n)$ is said to be weakly $T$-stable if $\{u_n\}$ converges to a fixed point $u^*$ of $T$, and if $\{v_n\}$ is a sequence in $U$ such that $\lim_{n\to\infty} \rho_b(v_{n+1}, f(T, v_n)) = 0$ and sequence $\{\rho_b(v_n, Tv_n)\}$ is bounded, then $\lim_{n\to\infty} v_n = u^*$.*

## Main Results

In this section, we will first give fixed point theorem for the Reich contraction mapping in complete convex $b$-metric spaces by means of Mann iteration scheme.

And also, we will show the weak $T$-stability of Mann iteration for this mapping in complete convex $b$-metric spaces. We begin with the following result:

**Theorem 1.1.** *Let $(U, \rho_b, \omega)$ be a complete convex $b$-metric space with constant $s > 1$ and $T : U \to U$ be a Reich contraction mapping defined by (1.1). Let $u_0 \in U$ such that $\rho_b(u_0, Tu_0) = M < \infty$ and define Mann iteration as $u_n = \omega(u_{n-1}, Tu_{n-1}; \alpha_{n-1})$ for $n \in \mathbb{N}$ and $\alpha_{n-1} \in (0, \frac{1}{4s^2}]$. If we get $k \in (0, \frac{1}{4s^2}]$ and $t \in (0, \frac{1}{8s^2}]$ for $0 \leq k + 2t < \frac{1}{2s^2}$, then $T$ has a unique fixed point in $U$.*

*Proof.* For any $n \in \mathbb{N}$, we have

$$\rho_b(u_n, u_{n+1}) = \rho_b(u_n, \omega(u_n, Tu_n; \alpha_n)) \leq (1 - \alpha_n)\rho_b(u_n, Tu_n) \tag{1.3}$$

and

$$\begin{aligned}
\rho_b(u_n, Tu_n) &= \rho_b(\omega(u_{n-1}, Tu_{n-1}; \alpha_{n-1}), Tu_n) \\
&\leq \alpha_{n-1}\rho_b(u_{n-1}, Tu_n) + (1 - \alpha_{n-1})\rho_b(Tu_{n-1}, Tu_n) \\
&\leq s\alpha_{n-1}[\rho_b(u_{n-1}, Tu_{n-1}) + \rho_b(Tu_{n-1}, Tu_n)] + \rho_b(Tu_{n-1}, Tu_n) \\
&= s\alpha_{n-1}\rho_b(u_{n-1}, Tu_{n-1}) + (1 + s\alpha_{n-1})\rho_b(Tu_{n-1}, Tu_n) \\
&\leq s\alpha_{n-1}\rho_b(u_{n-1}, Tu_{n-1}) \\
&\quad + (1 + s\alpha_{n-1})[k\rho_b(u_{n-1}, u_n) + t(\rho_b(u_{n-1}, Tu_{n-1}) + \rho_b(u_n, Tu_n))] \\
&\leq (s\alpha_{n-1} + (s\alpha_{n-1} + 1)t)\rho_b(u_{n-1}, Tu_{n-1}) \\
&\quad + (s\alpha_{n-1} + 1)k\rho_b(u_{n-1}, u_n) + (s\alpha_{n-1} + 1)t\rho_b(u_n, Tu_n) \\
&\leq (s\alpha_{n-1} + (s\alpha_{n-1} + 1)t)\rho_b(u_{n-1}, Tu_{n-1}) \\
&\quad + (s\alpha_{n-1} + 1)k(1 - \alpha_{n-1})\rho_b(u_{n-1}, Tu_{n-1}) + (s\alpha_{n-1} + 1)t\rho_b(u_n, Tu_n).
\end{aligned}$$

Thus, we have

$$[1 - (s\alpha_{n-1} + 1)t]\rho_b(u_n, Tu_n) \leq [s\alpha_{n-1} + (s\alpha_{n-1} + 1)(t + k(1 - \alpha_{n-1}))]\rho_b(u_{n-1}, Tu_{n-1}).$$

Since $(s\alpha_{n-1} + 1)t \leq \left(s\frac{1}{4s^2} + 1\right)\frac{1}{8s^2} < 1$, we have

$$\rho_b(u_n, Tu_n) \leq \frac{s\alpha_{n-1} + (s\alpha_{n-1} + 1)(t + k(1 - \alpha_{n-1}))}{1 - (s\alpha_{n-1} + 1)t}\rho_b(u_{n-1}, Tu_{n-1}). \tag{1.4}$$

Note that $\lambda_{n-1}=\frac{s\alpha_{n-1}+(s\alpha_{n-1}+1)(t+k(1-\alpha_{n-1}))}{1-(s\alpha_{n-1}+1)t}$ for $n\in\mathbb{N}$ and using $\alpha_{n-1}\in(0,\frac{1}{4s^2}]$, $k\in(0,\frac{1}{4s^2}]$ and $t\in(0,\frac{1}{8s^2}]$, we conclude that

$$\begin{aligned}\lambda_{n-1} &= \frac{s\alpha_{n-1}+(s\alpha_{n-1}+1)\,(t+k\,(1-\alpha_{n-1}))}{1-(s\alpha_{n-1}+1)\,t}\\ &\leq \frac{s\frac{1}{4s^2}+\left(s\frac{1}{4s^2}+1\right)\left(\frac{1}{8s^2}+\frac{1}{4s^2}\right)}{1-\left(s\frac{1}{4s^2}+1\right)\frac{1}{8s^2}}\\ &< \frac{23}{27}.\end{aligned}$$

From last inequality and inequality (1.4) with the assupmtions of the theorem, we obtain

$$\rho_b(u_n,Tu_n)\leq\lambda_{n-1}\rho_b(u_{n-1},Tu_{n-1})<\frac{23}{27}\rho_b(u_{n-1},Tu_{n-1}) \tag{1.5}$$

which implies that $\{\rho_b(u_n,Tu_n)\}$ is decreasing sequence of non-negative reals. Therefore, there exists $\delta\geq 0$, such that

$$\lim_{n\to\infty}\rho_b(u_n,Tu_n)=\delta.$$

We will show that $\delta=0$. Assume that $\delta>0$. From (1.5), for $n\to\infty$, we obtain

$$\delta<\frac{23}{27}\delta<\delta.$$

This is a contradiction. Hence, we have $\delta=0$ and mean that

$$\lim_{n\to\infty}\rho_b(u_n,Tu_n)=0.$$

Furthermore, by inequality (1.3), we get

$$\rho_b(u_n,u_{n+1})\leq(1-\alpha_n)\,\rho_b(u_n,Tu_n)<\rho_b(u_n,Tu_n).$$

This implies that $\lim_{n\to\infty}\rho_b(u_n,u_{n+1})=0$.

Now, we will show that $\{u_n\}$ is a Cauchy sequence. Suppose that $\{u_n\}$ is not a Cauchy sequence. Then, there exists $\varepsilon_0>0$ and subsequences $\left\{u_{\phi(l)}\right\}$ and $\left\{u_{\eta(l)}\right\}$

of $\{u_n\}$ such that $\phi(l)$ is the smallest naturel index with $\phi(l) > \eta(l) > l$,

$$\rho_b(u_{\phi(l)}, u_{\eta(l)}) \geq \varepsilon_0$$

and

$$\rho_b(u_{\phi(l)-1}, u_{\eta(l)}) < \varepsilon_0.$$

Then, we get

$$\varepsilon_0 \leq \rho_b(u_{\phi(l)}, u_{\eta(l)}) \leq s\left[\rho_b(u_{\phi(l)}, u_{\eta(l)+1}) + \rho_b(u_{\eta(l)+1}, u_{\eta(l)})\right],$$

which implies that

$$\frac{\varepsilon_0}{s} \leq \lim_{l\to\infty} \sup \rho_b(u_{\phi(l)}, u_{\eta(l)+1}).$$

Note that

$$\begin{aligned}
\rho_b(u_{\phi(l)}, u_{\eta(l)+1}) &= \rho_b\left(\omega\left(u_{\phi(l)-1}, Tu_{\phi(l)-1}; \alpha_{\phi(l)-1}\right), u_{\eta(l)+1}\right) \\
&\leq \alpha_{\phi(l)-1}\rho_b(u_{\phi(l)-1}, u_{\eta(l)+1}) + (1-\alpha_{\phi(l)-1})\rho_b(Tu_{\phi(l)-1}, u_{\eta(l)+1}) \\
&\leq \alpha_{\phi(l)-1}\rho_b(u_{\phi(l)-1}, u_{\eta(l)+1}) \\
&\quad +(1-\alpha_{\phi(l)-1})s\left[\rho_b(Tu_{\phi(l)-1}, Tu_{\eta(l)+1}) + \rho_b(Tu_{\eta(l)+1}, u_{\eta(l)+1})\right] \\
&\leq \alpha_{\phi(l)-1}s\left[\rho_b(u_{\phi(l)-1}, u_{\eta(l)}) + \rho_b(u_{\eta(l)}, u_{\eta(l)+1})\right] \\
&\quad +(1-\alpha_{\phi(l)-1})s\left[k\rho_b(u_{\phi(l)-1}, u_{\eta(l)+1}) + t\rho_b(u_{\phi(l)-1}, Tu_{\phi(l)-1})\right] \\
&\quad +(1-\alpha_{\phi(l)-1})s(t+1)\rho_b(u_{\eta(l)+1}, Tu_{\eta(l)+1}) \\
&\leq \left(\alpha_{\phi(l)-1}s + (1-\alpha_{\phi(l)-1})s^2k\right)\left[\rho_b(u_{\phi(l)-1}, u_{\eta(l)}) + \rho_b(u_{\eta(l)}, u_{\eta(l)+1})\right] \\
&\quad +(1-\alpha_{\phi(l)-1})s\left[t\rho_b(u_{\phi(l)-1}, Tu_{\phi(l)-1}) + (t+1)\rho_b(u_{\eta(l)+1}, Tu_{\eta(l)+1})\right].
\end{aligned}$$

From this inequality, we get

$$\lim_{l\to\infty} \sup \rho_b(u_{\phi(l)}, u_{\eta(l)+1}) \leq \left(\alpha_{\phi(l)-1}s + (1-\alpha_{\phi(l)-1})s^2k\right)\varepsilon_0 < \left(\frac{1}{4s^2}s + s^2\frac{1}{4s^2}\right)\varepsilon_0 < \frac{1}{2}\varepsilon_0,$$

which is a contradiction. Thus, $\{u_n\}$ is a Cauchy sequence in $U$. By the completeness of $U$, it follows that there exists $u^* \in U$ such that

$$\lim_{n\to\infty} \rho_b(u_n, u^*) = 0.$$

Now, we will show that $u^*$ is a fixed point of $T$. Since

$$\begin{aligned}\rho_b(u^*,Tu^*) &\leq s\left[\rho_b(u^*,u_n)+\rho_b(u_n,Tu^*)\right]\\ &\leq s\rho_b(u^*,u_n)+s^2\left[\rho_b(u_n,Tu_n)+\rho_b(Tu_n,Tu^*)\right]\\ &\leq s\rho_b(u^*,u_n)+s^2\rho_b(u_n,Tu_n)\\ &\quad +s^2\left[k\rho_b(u_n,u^*)+t\left(\rho_b(u_n,Tu_n)+\rho_b(u^*,Tu^*)\right)\right],\end{aligned}$$

we conclude that

$$\begin{aligned}(1-s^2t)\rho_b(u^*,Tu^*) &\leq (s+s^2k)\rho_b(u^*,u_n)+(s^2+s^2t)\rho_b(u_n,Tu_n)\\ &\leq (s+s^2k)\rho_b(u^*,u_n)+(s^2+s^2t)\left(\frac{23}{27}\right)^n\rho_b(u_0,Tu_0).\end{aligned}$$

Consequently, we get that $\lim_{n\to\infty}\rho_b(u^*,Tu^*)=0$, so $u^*$ is a fixed point of $T$. Let $v^*\in U$ be another fixed point of $T$ and assume that $v^*\neq u^*$. Then $Tv^*=v^*$. Then using (1.1), we get

$$\rho_b(u^*,v^*)=\rho_b(Tu^*,Tv^*)\leq k\rho_b(u^*,v^*)+t\left[\rho_b(u^*,Tu^*)+\rho_b(v^*,Tv^*)\right].$$

For $k\in[0,1)$,

$$(1-k)\rho_b(u^*,v^*)\leq 0$$

is a contradiction. Hence, $u^*=v^*$, which completes the proof. □

Now, we will give the following example that satisfies the conditions of our main Theorem 1.1.

**Example 1.1.** *Let $U=[0,\infty)$ and $Tu=\frac{u}{6}$ for all $u\in U$. For any $u,v\in U$, we define mapping $\rho_b:U\times U\to[0,\infty)$ by the formula $\rho\left(u,v\right)=\left(u-v\right)^2$, while the mapping $\omega:U\times U\times[0,1]\to U$ is defined as*

$$\omega\left(u,v;\alpha\right)\leq\alpha u+\left(1-\alpha\right)v$$

*for all $u,v\in U$ and all $\alpha\in[0,1]$. Set $u_n=\omega\left(u_{n-1},Tu_{n-1};\alpha_{n-1}\right)$ and $\alpha_{n-1}=\frac{1}{4s^2}$. Then $(U,\rho_b,\omega)$ is a complete convex b-metric space with $s=2$. Now, we choose $u_0\in U\setminus\{0\}$. Combining with $u_n=\omega\left(u_{n-1},Tu_{n-1};\alpha_{n-1}\right)$, $\alpha_{n-1}=$*

*$\frac{1}{4s^2} = \frac{1}{16}$ and $Tu = \frac{u}{6}$, we obtain*

$$\begin{aligned} u_n &= \omega\left(u_{n-1}, Tu_{n-1}; \alpha_{n-1}\right) = \alpha_{n-1}u_{n-1} + \left(1-\alpha_{n-1}\right)Tu_{n-1} \\ &= \frac{1}{16}u_{n-1} + \left(1 - \frac{1}{16}\right)\frac{u_{n-1}}{6} \\ &= \frac{7}{32}u_{n-1}. \end{aligned}$$

*Similarly, we have*

$$u_{n-1} = \frac{7}{32}u_{n-2},\ u_{n-2} = \frac{7}{32}u_{n-3}, \cdots,\ u_1 = \frac{7}{32}u_0.$$

*Therefore,*

$$u_n = \left(\frac{7}{32}\right)^n u_0 \text{ and } Tu_n = \frac{1}{6}\left(\frac{7}{32}\right)^n u_0.$$

*If we take limits of above sequences as $n \to \infty$, we have that $u_n \to 0$ and $Tu_n \to 0$. That is $0$ is a fixed point of $T$.*

**Theorem 1.2.** *Let $(U, \rho_b, \omega)$ be a complete convex $b$-metric space with constant $s > 1$ and $T : U \to U$ be a Reich contraction mapping defined by (1.1). Let $u_0 \in U$ such that $\rho_b(u_0, Tu_0) = M < \infty$ and define Mann iteration as $u_n = \omega\left(u_{n-1}, Tu_{n-1}; \alpha_{n-1}\right)$ for $n \in \mathbb{N}$ and $\alpha_{n-1} \in (0, \frac{1}{4s^2}]$. If we get $k \in (0, \frac{1}{16s^2}]$ and $t \in (0, \frac{1}{4s^2}]$ for $0 \leq k + 2t < \frac{9}{16s^2}$, then $T$ has a unique fixed point in $U$.*

*Proof.* The proof is similar to the proof of Theorem 1.1. □

After our theorems, we conclude that followings:

**Remark 1.1.** *(i) If we take $t = 0$ in Theorem 1.1, Reich contraction mapping is deduced contraction mapping and Theorem 1 in [4] is obtained.*

*(ii) If we take $k = 0$ in Theorem 1.2, Reich contraction mapping is deduced Kannan contraction mapping and Theorem 2 in [4] is obtained.*

Next, we will consider the problem for the weak $T-$stability of Mann iteration process for the Reich contraction mapping in complete convex $b$-metric spaces.

**Theorem 1.3.** *Under the assumptions of Theorem 1.1, additionally, if $\lim_{n\to\infty} \alpha_n = 0$ and if condition $\frac{s^2(k+ts)}{1-ts} < 1$, then Mann iteration is weakly $T$-stable.*

*Proof.* From Theorem 1.1, it follows that $u^*$ is a unique fixed point of $T$ in $U$. Suppose that $\{v_n\}$ is a sequence in $U$ and satisfies $\lim_{n\to\infty}\rho_b(v_{n+1},\omega(v_n,Tv_n;\alpha_n))=0$ and $\{\rho_b(v_n,Tv_n)\}$ is bounded. We get

$$\begin{aligned}\rho_b(v_{n+1},u^*) &\leq s\left[\rho_b(v_{n+1},\omega(v_n,Tv_n;\alpha_n))+\rho_b(\omega(v_n,Tv_n;\alpha_n),u^*)\right]\\ &\leq s\rho_b(v_{n+1},\omega(v_n,Tv_n;\alpha_n))+s^2\left[\rho_b(\omega(v_n,Tv_n;\alpha_n),Tv_n)+\rho_b(Tv_n,u^*)\right]\\ &\leq s\rho_b(v_{n+1},\omega(v_n,Tv_n;\alpha_n))+s^2\alpha_n\rho_b(v_n,Tv_n)+s^2\rho_b(Tv_n,u^*).\end{aligned}$$

Note that

$$\begin{aligned}\rho_b(Tv_n,u^*) &\leq k\rho_b(v_n,u^*)+t\left[\rho_b(v_n,Tv_n)+\rho_b(u^*,Tu^*)\right]\\ &\leq k\rho_b(v_n,u^*)+ts\left[\rho_b(v_n,u^*)+\rho_b(u^*,Tv_n)\right].\end{aligned}$$

Therefore, we get $\rho_b(Tv_n,u^*)\leq\frac{k+ts}{1-ts}\rho_b(v_n,u^*)$. Using this inequality, we obtain

$$\rho_b(v_{n+1},u^*)\leq s\rho_b(v_{n+1},\omega(v_n,Tv_n;\alpha_n))+s^2\alpha_n\rho_b(v_n,Tv_n)+\frac{s^2(k+ts)}{1-ts}\rho_b(v_n,u^*).$$

Considering the conditions of the theorem and from Lemma 1.1, we get that

$$\lim_{n\to\infty}\rho_b(v_n,u^*)=0,$$

which completes the proof. □

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