Volume 37, Numbers 1-2 (2018)

THE ALIGARH BULLETIN OF MATHEMATICS

Department of Mathematics
Aligarh Muslim University
Aligarh

THE ALIGARH BULLETIN OF MATHEMATICS

CONTENTS

Vol. 37 Numbers 1-2 (2018)

The Aligarh Bulletin of Mathematics
Volume 37, Numbers 1-2 (2018), 1-20

ISSN: 0304-9787

Relative $(p, q, t)L$-th order, relative $(p, q, t)L$-th type and relative (p, q, t)L-th weak type based some growth analysis of composite analytic functions of several complex variables in the unit polydisc

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(Received September 12, 2018)

Abstract

In this paper, the ideas of relative (p, q, t) L-th order, relative (p, q, t) L-th lower order, relative (p, q, t) L-th type and relative (p, q, t) L-th weak type of an analytic function with respect to an entire function in the unit polydisc are introduced. Hence we study some comparative growth properties of composition of two analytic functions in the unit polydisc on the basis of relative (p, q, t) L-th order, relative (p, q, t) L-th lower order, relative (p, q, t) L-th type and relative (p, q, t) L-th weak type where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$.

1 Introduction, Definitions and Notations

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in

Keywords and phrases : Growth, analytic function, composition, unit polydisc, relative (p, q, t) L-th order, relative (p, q, t) L-th lower order, relative (p, q, t) L-th type, relative (p, q, t) L-th lower type, relative (p, q, t) L-th weak type, slowly changing function in the unit polydisc
2010 AMS Subject Classification : 32A15

the unit disc $U = \{z : |z| < 1\}$ and $M_f(r)$ be the maximum of $|f(z)|$ on $|z| = r$. In [8], Sons was define the order $\rho(f)$ and the lower order $\lambda(f)$ as

$$\rho(f) = \overline{\lim_{r\to 1}} \frac{\log^{[2]} M_f(r)}{-\log(1-r)} \text{ and } \lambda(f) = \underline{\lim_{r\to 1}} \frac{\log^{[2]} M_f(r)}{-\log(1-r)}.$$

Considering the unit polydisc $U = \{(z_1, z_2) : |z_j| \le 1, j = 1, 2\}$, Banerjee and Dutta [2] introduced the definition of order and lower order of functions of two complex variables analytic in the unit polydisc which are as follows:

Definition 1. *Let $f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n$ be a non-constant analytic function of two complex variables z_1 and z_2 holomorphic in the closed unit polydisc $U = \{(z_1, z_2) : |z_j| \le 1,\ j = 1, 2\}$ and $M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_j| \le 1, j = 1, 2\}$. Then order and lower order of $f(z_1, z_2)$ are denoted by $\rho(f)$ and $\lambda(f)$ respectively and defined as*

$${}_{v_2}\rho(f) = \overline{\lim_{r_1,r_2\to 1}} \frac{\log^{[2]} M_f(r_1, r_2)}{-\log(1-r_1)(1-r_2)} \text{ and } {}_{v_2}\lambda(f) = \underline{\lim_{r_1,r_2\to 1}} \frac{\log^{[2]} M_f(r_1, r_2)}{-\log(1-r_1)(1-r_2)}.$$

Generalizing this notion, Dutta [5] introduced the definitions of (p, q)-th order and lower (p, q)-th lower order of functions of two complex variables analytic in the unit polydisc in the following way:

Definition 2. *[5] Let $f(z_1, z_2)$ be a non-constant analytic function of two complex variables z_1 and z_2 holomorphic in the closed unit polydisc $U = \{(z_1, z_2) : |z_j| \le 1, j = 1, 2\}$. then (p, q)-th order ${}_{v_2}\rho^{(p,q)}(f)$ and lower (p, q)-th lower order ${}_{v_2}\lambda^{(p,q)}(f)$ of $f(z_1, z_2)$ are defined by*

$${}_{v_2}\rho^{(p,q)}(f) = \overline{\lim_{r_1,r_2\to 1}} \frac{\log^{[p]} M_f(r_1, r_2)}{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)} \text{ and } {}_{v_2}\lambda^{(p,q)}(f) = \underline{\lim_{r_1,r_2\to 1}} \frac{\log^{[p]} M_f(r_1, r_2)}{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)}$$

where where p and q are positive integers with $p \ge q \ge 1$.

Extending this notion, one may introduce (p, q)-th order and lower (p, q)-th lower order for functions of n-complex variables analytic in a unit polydisc as follows :

$${}_{v_n}\rho^{(p,q)}(f) = \overline{\lim_{r_1,r_2,\cdots,r_n\to 1}} \frac{\log^{[p]} M_f(r_1, r_2, \cdot\cdot, r_n)}{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right)}$$

and

$${}_{v_n}\lambda^{(p,q)}(f) = \underline{\lim_{r_1,r_2,\cdots,r_n\to 1}} \frac{\log^{[p]} M_f(r_1, r_2, \cdot\cdot, r_n)}{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right)}$$

where $p,\ q \in \mathbb{N}$ and $f(z_1, z_2, \cdot\cdot\cdot, z_n)$ be a non-constant analytic function of n-complex variables $z_1, z_2, \cdots, z_{n-1}$ and z_n in the unit polydisc

$$U = \{(z_1, z_2, \cdot\cdot, z_n) : |z_j| \leq 1, j = 1, 2, \cdot\cdot, n; r_1 > 0, r_2 > 0, \cdot\cdot, r_n > 0\}$$

and

$$M_f (r_1, r_2, \cdot\cdot, r_n) = \max \{|f(z_1, z_2, \cdot\cdot, z_n)| : |z_j| \leq 1, j = 1, 2, \cdot\cdot, n; r_1 > 0, r_2 > 0, \cdot\cdot, r_n > 0\}.$$

The above definition avoids the restriction $p \geq q \geq 1$ of the original definition of (p, q)-th order (respectively (p, q)-th lower order) of functions of two complex variables holomorphic in the unit polydisc as introduced by Dutta [5].

In this connection we just recall the following definition :

Definition 3. *A a non-constant analytic function of n-complex variables $f(z_1, z_2, \cdots, z_n)$ is said to have index-pair (p, q) if $b < {}_{v_n}\rho^{(p,q)}(f) < \infty$ and ${}_{v_n}\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < {}_{v_n}\rho^{(p,q)}(f) < \infty$, then*

$$\begin{cases} {}_{v_n}\rho^{(p-n,q)}(f) = \infty \text{ for } & n < p, \\ {}_{v_n}\rho^{(p,q-n)}(f) = 0 \text{ for } & n < q, \\ {}_{v_n}\rho^{(p+n,q+n)}(f) = 1 \text{ for } & n = 1, 2, \ldots \end{cases}.$$

Similarly for $0 < {}_{v_n}\lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} {}_{v_n}\lambda^{(p-n,q)}(f) = \infty \text{ for } & n < p, \\ {}_{v_n}\lambda^{(p,q-n)}(f) = 0 \text{ for } & n < q, \\ {}_{v_n}\lambda^{(p+n,q+n)}(f) = 1 \text{ for } & n = 1, 2, \ldots \end{cases}.$$

The function $f(z_1, z_2, \cdots, z_n)$ is said to be of regular (p, q) growth when (p, q)-th order and (p, q)-th lower order of $f(z_1, z_2, \cdots, z_n)$ are the same. Functions which are not of regular (p, q) growth are said to be of irregular (p, q)-growth.

Somasundaram and Thamizharasi [9] introduced the notions of L-order (L-lower order) for entire functions of single variable where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. In the line of Somasundaram and Thamizharasi [9] one may introduce the definition of $(p, q, t)L$-th order and $(p, q, t)L$-th lower order for functions of n complex variables holomorphic in a unit polydisc in the following way:

$${}_{v_n}\rho^{(p,q,t)L}(f) = \varlimsup_{r_1, r_2, \cdots, r_n \to 1} \frac{\log^{[p]} M_f (r_1, r_2, \cdot\cdot, r_n)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right) + \exp^{[t]} L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)}$$

and

$$ {}_{v_n}\lambda^{(p,q,t)L}(f) = \underline{\lim}_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p]} M_f(r_1,r_2,\cdot\cdot,r_n)}{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right) + \exp^{[t]} L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)}, $$

where $p,\ q \in \mathbb{N}$, $t \in \mathbb{N}\cup\{-1,0\}$ and $L \equiv L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdot\cdot,\frac{1}{1-r_n}\right)$ is a positive continuous function in the unit polydisc U increasing slowly i.e., $L\left(\frac{a}{1-r_1},\frac{a}{1-r_2},\cdot\cdot,\frac{a}{1-r_n}\right) \sim L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdot\cdot,\frac{1}{1-r_n}\right)$ as $r\to 1$, for every positive constant 'a'.

Mainly the growth investigation of analytic functions of single variable has usually been done through their maximum moduli in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any analytic function with respect to a new one, the notions of relative growth indicators [3, 4] will come. Considering this notion one may introduce the definition of relative (p,q)-th order and relative (p,q)-th lower order in the unit polydisc which are as follows:

Definition 4. *If $f(z_1,z_2,\cdot\cdot,z_n)$ be holomorphic in U and $g(z_1,z_2,\cdot\cdot,z_n)$ be entire function of n-complex variables, then the relative (p,q)-th order $f(z_1,z_2,\cdot\cdot,z_n)$ with respect to $g(z_1,z_2,\cdot\cdot,z_n)$, denoted by ${}_{v_n}\rho_g^{(p,q)}(f)$ is defined by*

$$ {}_{v_n}\rho_g^{(p,q)}(f) = \overline{\lim}_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p]} M_g^{-1}\left(M_f(r_1,r_2,\cdot\cdot,r_n)\right)}{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right)}, $$

where p and q are any two positive integers.

Similarly for any two positive integers p and q, the relative (p,q)-th lower order of $f(z_1,z_2,\cdot\cdot,z_n)$ with respect to $g(z_1,z_2,\cdot\cdot,z_n)$, denoted by ${}_{v_n}\lambda_g^{(p,q)}(f)$ is given by

$$ {}_{v_n}\lambda_g^{(p,q)}(f) = \underline{\lim}_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p]} M_g^{-1}\left(M_f(r_1,r_2,\cdot\cdot,r_n)\right)}{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right)}. $$

In order to make some progress in the study of relative order in the unit polydisc, now we introduce relative $(p,q,t)\,L$-th order and relative $(p,q,t)\,L$-th lower order in the following way:

Definition 5. *If $f(z_1,z_2,\cdot\cdot,z_n)$ be holomorphic in U and $g(z_1,z_2,\cdot\cdot,z_n)$ be entire function of n-complex variables, then the relative $(p,q,t)L$-th order denoted as $\rho_g^{(p,q,t)L}(f)$ and relative $(p,q,t)L$-th lower order denoted as $\lambda_g^{(p,q,t)L}(f)$ of $f(z_1,z_2,\cdot\cdot,z_n)$ with respect to $g(z_1,z_2,\cdot\cdot,z_n)$ are define by*

$$ {}_{v_n}\rho_g^{(p,q,t)L}(f) = \overline{\lim}_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p]} M_g^{-1}\left(M_f(r_1,r_2,\cdot\cdot,r_n)\right)}{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right) + \exp^{[t]} L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdot\cdot,\frac{1}{1-r_n}\right)} $$

and

$$_{v_n}\lambda_g^{(p,q,t)L}(f) = \varliminf_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p]} M_g^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right) + \exp^{[t]} L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdot\cdot,\frac{1}{1-r_n}\right)},$$

where $p,q\in\mathbb{N}$ *and* $t\in\mathbb{N}\cup\{-1,0\}$.

Now to compare the relative growth of two analytic functions in the unit polydisc having same non zero finite relative (p,q,t) L-th order with respect to another entire function of n-complex variables, one may introduce the concepts of relative (p,q,t) L-th type and relative (p,q,t) L-th lower type in the unit polydisc in the following manner:

Definition 6. *Let $f(z_1,z_2,\cdot\cdot,z_n)$ be holomorphic in U and $g(z_1,z_2,\cdot\cdot,z_n)$ be entire function of n-complex variables with $0 < {}_{v_n}\rho_g^{(p,q,t)L}(f) < \infty$, then the relative (p,q,t) L-th type and relative (p,q,t) L-th lower type denoted respectively by ${}_{v_n}\sigma_g^{(p,q,t)L}(f)$ and ${}_{v_n}\overline{\sigma}_g^{(p,q,t)L}(f)$ of $f(z_1,z_2,\cdot\cdot,z_n)$ with respect to $g(z_1,z_2,\cdot\cdot,z_n)$ are respectively defined as follows:*

$$_{v_n}\sigma_g^{(p,q,t)L}(f) =$$

$$\varlimsup_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p-1]} M_g^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\left[\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right)\cdot\exp^{[t+1]} L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdot\cdot,\frac{1}{1-r_n}\right)\right]^{{}_{v_n}\rho_g^{(p,q,t)L}(f)}}$$

and

$$_{v_n}\overline{\sigma}_g^{(p,q,t)L}(f) =$$

$$\varliminf_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p-1]} M_g^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\left[\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right)\cdot\exp^{[t+1]} L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdot\cdot,\frac{1}{1-r_n}\right)\right]^{{}_{v_n}\rho_g^{(p,q,t)L}(f)}},$$

where $p,q\in\mathbb{N}$ *and* $t\in\mathbb{N}\cup\{-1,0\}$.

Analogously to determine the relative growth of two analytic functions in the unit polydisc having same non zero finite relative (p,q,t) L-th lower order with respect to another entire function of n-complex variables, one may introduce the definition of relative (p,q,t) L-th weak type in the unit polydisc in the following way:

Definition 7. *Let $f(z_1,z_2,\cdot\cdot,z_n)$ be holomorphic in U and $g(z_1,z_2,\cdot\cdot,z_n)$ be entire function of n-complex variables with $0 < {}_{v_n}\lambda_g^{(p,q,t)L}(f) < \infty$, then the relative (p,q,t) L-th weak type denoted by ${}_{v_n}\tau_g^{(p,q,t)L}(f)$ of $f(z_1,z_2,\cdot\cdot,z_n)$ with respect to $g(z_1,z_2,\cdot\cdot,z_n)$ is defined as follows:*

$$_{v_n}\tau_g^{(p,q,t)L}(f) =$$

$$\varliminf_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p-1]} M_g^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\left[\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right)\cdot \exp^{[t+1]} L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdot\cdot,\frac{1}{1-r_n}\right)\right]^{{}_{v_n}\lambda_g^{(p,q,t)L}(f)}}$$

where $p,q\in\mathbb{N}$ *and* $t\in\mathbb{N}\cup\{-1,0\}$.

Also one may define the growth indicator ${}_{v_n}\overline{\tau}_g^{(p,q,t)L}(f)$ *in the unit polydisc in the following manner :*

$${}_{v_n}\overline{\tau}_g^{(p,q,t)L}(f) =$$

$$\varlimsup_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p-1]} M_g^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\left[\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right)\cdot \exp^{[t+1]} L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdot\cdot,\frac{1}{1-r_n}\right)\right]^{{}_{v_n}\lambda_g^{(p,q,t)L}(f)}}$$

where $p,q\in\mathbb{N}$ *and* $t\in\mathbb{N}\cup\{-1,0\}$.

In this paper we study some growth properties relating to the composition of two analytic function of n-complex variables in the unit polydisc on the basis of relative (p,q,t) L-th type, relative (p,q,t) L-th lower type and relative (p,q,t) L-th weak type as compared to the growth of their corresponding left and right factors where $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$. We do not explain the standard definitions and notations in the theory of entire functions as those are available in 1,

2 Theorems

In this section we present the main results of the paper.

Theorem 1. *If* $f(z_1,z_2,\cdot\cdot,z_n)$, $g(z_1,z_2,\cdot\cdot,z_n)$ *be any two analytic functions in* U *and* $h(z_1,z_2,\cdot\cdot,z_n)$ *be an entire function of* n*-complex variables such that* $0<{}_{v_n}\lambda_h^{(p,q,t)L}(f\circ g)\le {}_{v_n}\rho_h^{(p,q,t)L}(f\circ g)<\infty$ *and* $0<{}_{v_n}\lambda_h^{(p,q,t)L}(f)\le {}_{v_n}\rho_h^{(p,q,t)L}(f)<\infty$, *where* $p,q\in\mathbb{N}$ *and* $t\in\mathbb{N}\cup\{-1,0\}$. *Then*

$$\begin{aligned}
\frac{{}_{v_n}\lambda_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\rho_h^{(p,q,t)L}(f)} &\le \varliminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}\\
&\le \min\left\{\frac{{}_{v_n}\lambda_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\lambda_h^{(p,q,t)L}(f)},\frac{{}_{v_n}\rho_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\rho_h^{(p,q,t)L}(f)}\right\}\\
&\le \max\left\{\frac{{}_{v_n}\lambda_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\lambda_h^{(p,q,t)L}(f)},\frac{{}_{v_n}\rho_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\rho_h^{(p,q,t)L}(f)}\right\}\\
&\le \varlimsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}\le\frac{{}_{v_n}\rho_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\lambda_h^{(p,q,t)L}(f)}\,.
\end{aligned}$$

Proof. From the definition of ${}_{v_n}\rho_h^{(p,q,t)L}(f)$ and ${}_{v_n}\lambda_h^{(p,q,t)L}(f\circ g)$, we have for arbitrary positive ε and for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ that

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right) \geqslant \left({}_{v_n}\lambda_h^{(p,q,t)L}\left(f\circ g\right)-\varepsilon\right)\times$$
$$\left[\log^{[q]}\left(\frac{1}{\left(1-r_1\right)\left(1-r_2\right)\cdot\cdot\left(1-r_n\right)}\right)+\exp^{[t]} L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdot\cdot, \frac{1}{1-r_n}\right)\right] \quad (2.1)$$

and

$$\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right) \leq \left({}_{v_n}\rho_h^{(p,q,t)L}\left(f\right)+\varepsilon\right)\times$$
$$\left[\log^{[q]}\left(\frac{1}{\left(1-r_1\right)\left(1-r_2\right)\cdot\cdot\left(1-r_n\right)}\right)+\exp^{[t]} L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdot\cdot, \frac{1}{1-r_n}\right)\right]. \quad (2.2)$$

Now from (2.1) and (2.2) it follows for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ that

$$\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \geqslant \frac{\left({}_{v_n}\lambda_h^{(p,q,t)L}\left(f\circ g\right)-\varepsilon\right)}{\left({}_{v_n}\rho_h^{(p,q,t)L}\left(f\right)+\varepsilon\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\varliminf_{r_1, r_2, \cdots, r_n\to 1}\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \geqslant \frac{{}_{v_n}\lambda_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\rho_h^{(p,q,t)L}\left(f\right)}. \quad (2.3)$$

Again for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity,

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right) \leq \left({}_{v_n}\lambda_h^{(p,q,t)L}\left(f\circ g\right)+\varepsilon\right)\times$$
$$\left[\log^{[q]}\left(\frac{1}{\left(1-r_1\right)\left(1-r_2\right)\cdot\cdot\left(1-r_n\right)}\right)+\exp^{[t]} L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdot\cdot, \frac{1}{1-r_n}\right)\right] \quad (2.4)$$

and for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$,

$$\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right) \geqslant \left({}_{v_n}\lambda_h^{(p,q,t)L}\left(f\right)-\varepsilon\right)\times$$
$$\log\left[\log^{[q]}\left(\frac{1}{\left(1-r_1\right)\left(1-r_2\right)\cdot\cdot\left(1-r_n\right)}\right)+\exp^{[t]} L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdot\cdot, \frac{1}{1-r_n}\right)\right]. \quad (2.5)$$

Combining (2.4) and (2.5), we get for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and

$\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \leq \frac{\left({}_{v_n}\lambda_h^{(p,q,t)L}\left(f\circ g\right)+\varepsilon\right)}{\left({}_{v_n}\lambda_h^{(p,q,t)L}\left(f\right)-\varepsilon\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\varliminf_{r_1, r_2, \cdots, r_n \to 1} \frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \leq \frac{{}_{v_n}\lambda_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\lambda_h^{(p,q,t)L}\left(f\right)}. \tag{2.6}$$

Also for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right) \leq \left({}_{v_n}\lambda_h^{(p,q,t)L}\left(f\right)+\varepsilon\right) \times$$
$$\left[\log^{[q]}\left(\frac{1}{\left(1-r_1\right)\left(1-r_2\right)\cdot\cdot\left(1-r_n\right)}\right)+\exp^{[t]} L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdot\cdot, \frac{1}{1-r_n}\right)\right]. \tag{2.7}$$

Now from (2.1) and (2.7), we obtain for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \geq \frac{\left({}_{v_n}\lambda_h^{(p,q,t)L}\left(f\circ g\right)-\varepsilon\right)}{\left({}_{v_n}\lambda_h^{(p,q,t)L}\left(f\right)+\varepsilon\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\varlimsup_{r_1, r_2, \cdots, r_n \to 1} \frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \geq \frac{{}_{v_n}\lambda_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\lambda_h^{(p,q,t)L}\left(f\right)}. \tag{2.8}$$

Also for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$,

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right) \leq \left({}_{v_n}\rho_h^{(p,q,t)L}\left(f\circ g\right)+\varepsilon\right) \times$$
$$\left[\log^{[q]}\left(\frac{1}{\left(1-r_1\right)\left(1-r_2\right)\cdot\cdot\left(1-r_n\right)}\right)+\exp^{[t]} L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdot\cdot, \frac{1}{1-r_n}\right)\right]. \tag{2.9}$$

Now it follows from (2.5) and (2.9) for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$

and $\left(\frac{1}{1-r_n}\right)$ that

$$\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \leq \frac{\left({}_{v_n}\rho_h^{(p,q,t)L}\left(f\circ g\right)+\varepsilon\right)}{\left({}_{v_n}\lambda_h^{(p,q,t)L}\left(f\right)-\varepsilon\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\overline{\lim_{r_1, r_2, \cdots, r_n \to 1}} \frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \leq \frac{{}_{v_n}\rho_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\lambda_h^{(p,q,t)L}\left(f\right)}. \tag{2.10}$$

Again from the definition of ${}_{v_n}\rho_h^{(p,q,t)L}\left(f\right)$, we get for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right) \geqslant \left({}_{v_n}\rho_h^{(p,q,t)L}\left(f\right)-\varepsilon\right)\times$$
$$\left[\log^{[q]}\left(\frac{1}{\left(1-r_1\right)\left(1-r_2\right)\cdot\cdot\left(1-r_n\right)}\right)+\exp^{[t]} L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdot\cdot, \frac{1}{1-r_n}\right)\right]. \tag{2.11}$$

Now from (2.9) and (2.11), it follows for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \leq \frac{\left({}_{v_n}\rho_h^{(p,q,t)L}\left(f\circ g\right)+\varepsilon\right)}{\left({}_{v_n}\rho_h^{(p,q,t)L}\left(f\right)-\varepsilon\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\underline{\lim_{r_1, r_2, \cdots, r_n \to 1}} \frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \leq \frac{{}_{v_n}\rho_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\rho_h^{(p,q,t)L}\left(f\right)}. \tag{2.12}$$

Again for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity,

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right) \geqslant \left({}_{v_n}\rho_h^{(p,q,t)L}\left(f\circ g\right)-\varepsilon\right)\times$$
$$\left[\log^{[q]}\left(\frac{1}{\left(1-r_1\right)\left(1-r_2\right)\cdot\cdot\left(1-r_n\right)}\right)+\exp^{[t]} L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdot\cdot, \frac{1}{1-r_n}\right)\right]. \tag{2.13}$$

So combining (2.2) and (2.13), we get for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$

and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \geqslant \frac{\left({}_{v_n}\rho_h^{(p,q,t)L}\left(f\circ g\right)-\varepsilon\right)}{\left({}_{v_n}\rho_h^{(p,q,t)L}\left(f\right)+\varepsilon\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\overline{\lim_{r_1,r_2,\cdots,r_n\to 1}} \frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \geqslant \frac{{}_{v_n}\rho_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\rho_h^{(p,q,t)L}\left(f\right)}. \tag{2.14}$$

Thus the theorem follows from (2.3), (2.6), (2.8), (2.10), (2.12) and (2.14). □

The following theorem can be proved in the line of Theorem 5 and so its proof is omitted.

Theorem 2. *If $f(z_1,z_2,\cdot\cdot,z_n)$, $g\left(z_1,z_2,\cdot\cdot,z_n\right)$ be any two analytic functions in U and $h\left(z_1,z_2,\cdot\cdot,z_n\right)$ be an entire function of n-complex variables such that $0<{}_{v_n}\lambda_h^{(p,q,t)L}\left(f\circ g\right)$ $\leq{}_{v_n}\rho_h^{(p,q,t)L}\left(f\circ g\right)<\infty$ and $0<{}_{v_n}\lambda_h^{(p,q,t)L}\left(g\right)\leq{}_{v_n}\rho_h^{(p,q,t)L}\left(g\right)<\infty$, where $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$. Then*

$$\begin{aligned}\frac{{}_{v_n}\lambda_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\rho_h^{(p,q,t)L}\left(g\right)} &\leq \underline{\lim_{r_1,r_2,\cdots,r_n\to 1}} \frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_g\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}\\ &\leq \min\left\{\frac{{}_{v_n}\lambda_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\lambda_h^{(p,q,t)L}\left(g\right)},\frac{{}_{v_n}\rho_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\rho_h^{(p,q,t)L}\left(g\right)}\right\}\\ &\leq \max\left\{\frac{{}_{v_n}\lambda_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\lambda_h^{(p,q,t)L}\left(g\right)},\frac{{}_{v_n}\rho_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\rho_h^{(p,q,t)L}\left(g\right)}\right\}\\ &\leq \overline{\lim_{r_1,r_2,\cdots,r_n\to 1}} \frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_g\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \leq \frac{{}_{v_n}\rho_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\lambda_h^{(p,q,t)L}\left(g\right)}.\end{aligned}$$

Theorem 3. *Let $f(z_1,z_2,\cdot\cdot,z_n)$, $g\left(z_1,z_2,\cdot\cdot,z_n\right)$ be any two analytic functions in U and $h\left(z_1,z_2,\cdot\cdot,z_n\right)$ be an entire function of n-complex variables such that ${}_{v_n}\lambda_h^{(p,q,t)L}\left(f\circ g\right)=$ ∞ and ${}_{v_n}\rho_h^{(p,q,t)L}\left(f\right)<\infty$ where $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$. Then*

$$\lim_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}=\infty\,.$$

Proof. If possible, let there exists a constant β such that for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, $\cdot\cdot$ $\cdot$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity we have

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)<\beta\cdot\log^{[p]} M_h^{-1}M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\,. \tag{2.15}$$

Again from the definition of ${}_{v_n}\rho_h^{(p,q,t)L}(f)$, it follows for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ that

$$\log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdots, r_n\right)\right) \leq \left({}_{v_n}\rho_h^{(p,q,t)L}(f) + \varepsilon\right) \times$$
$$\left[\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right) + \exp^{[t]} L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right]. \tag{2.16}$$

Now combining (2.15) and (2.16) we obtain for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdots, r_n\right)\right) \leq \beta \cdot \left({}_{v_n}\rho_h^{(p,q,t)L}(f) + \varepsilon\right) \times$$
$$\left[\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right) + \exp^{[t]} L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right]$$
$$i.e.,\ {}_{v_n}\lambda_h^{(p,q,t)L}(f\circ g) \leq \beta \cdot \left({}_{v_n}\rho_h^{(p,q,t)L}(f) + \varepsilon\right),$$

which contradicts the condition ${}_{v_n}\lambda_h^{(p,q,t)L}(f\circ g) = \infty$. So for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ we get that

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdots, r_n\right)\right) \geq \beta \cdot \log^{[p]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdots, r_n\right)\right),$$

from which the theorem follows. □

In the line of Theorem 3, one can easily prove the following theorem and therefore its proof is omitted.

Theorem 4. *Let $f(z_1, z_2, \cdots, z_n)$, $g(z_1, z_2, \cdots, z_n)$ be any two analytic functions in U and $h(z_1, z_2, \cdots, z_n)$ be an entire function of n-complex variables such that ${}_{v_n}\lambda_h^{(p,q,t)L}(f\circ g) = \infty$ and ${}_{v_n}\rho_h^{(p,q,t)L}(g) < \infty$ where $p, q \in \mathbb{N}$ and $t \in \mathbb{N}\cup\{-1, 0\}$. Then*

$$\lim_{r_1, r_2, \cdots, r_n \to 1} \frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdots, r_n\right)\right)}{\log^{[p]} M_h^{-1}\left(M_g\left(r_1, r_2, \cdots, r_n\right)\right)} = \infty.$$

Remark 1. *Theorem 3 is also valid with "limit superior" instead of "limit" if ${}_{v_n}\lambda_h^{(p,q,t)L}(f\circ g) = \infty$ is replaced by ${}_{v_n}\rho_h^{(p,q,t)L}(f\circ g) = \infty$ and the other conditions remain the same.*

Remark 2. *Theorem 4 is also valid with "limit superior" instead of "limit" if $\lambda_h^{(p,q,t)L}(f\circ g) = \infty$ is replaced by $\rho_h^{(p,q,t)L}(f\circ g) = \infty$ and the other conditions remain the same.*

Corollary 1. *Under the assumptions of Theorem 3 and Remark 1,*

$$\lim_{r_1,r_2,\cdots,r_n\to 1}\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}=\infty$$

and

$$\overline{\lim}_{r_1,r_2,\cdots,r_n\to 1}\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}=\infty$$

respectively.

Proof. By Theorem 3 we obtain for all sufficiently large values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdots$ and $\left(\frac{1}{1-r_n}\right)$ and for $K>1$,

$$\begin{aligned}\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right) &\geq K\cdot\log^{[p]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)\\ i.e.,\ \log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right) &\geq \left\{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)\right\}^K,\end{aligned}$$

from which the first part of the corollary follows.

Similarly using Remark 1, we obtain the second part of the corollary. □

Corollary 2. *Under the assumptions of Theorem 4 and Remark 2,*

$$\lim_{r_1,r_2,\cdots,r_n\to 1}\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_g\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}=\infty$$

and

$$\overline{\lim}_{r_1,r_2,\cdots,r_n\to 1}\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_g\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}=\infty$$

respectively.

In the line of Corollary 1, one can easily verify Corollary 2 with the help of Theorem 4 and Remark 2 respectively and therefore its proof is omitted.

Theorem 5. *If $f(z_1,z_2,\cdot\cdot,z_n)$, $g\left(z_1,z_2,\cdot\cdot,z_n\right)$ be any two analytic functions in U and $h\left(z_1,z_2,\cdot\cdot,z_n\right)$ be an entire function of n-complex variables such that $0<{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}\left(f\circ g\right)\leq{}_{v_n}\sigma_h^{(p,q,t)L}\left(f\circ g\right)<\infty$, $0<{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}\left(f\right)\leq{}_{v_n}\sigma_h^{(p,q,t)L}\left(f\right)<\infty$ and ${}_{v_n}\rho_h^{(p,q,t)L}\left(f\circ g\right)={}_{v_n}\rho_h^{(p,q,t)L}\left(f\right)$, where $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$. Then*

$$\begin{aligned}\frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\sigma_h^{(p,q,t)L}\left(f\right)} &\leq \underline{\lim}_{r_1,r_2,\cdots,r_n\to 1}\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}\\ &\leq \min\left\{\frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}\left(f\right)},\frac{{}_{v_n}\sigma_h^{(p,q,t)L}\left(f\circ g\right)}{{}_{v_n}\sigma_h^{(p,q,t)L}\left(f\right)}\right\}\end{aligned}$$

$$\leq \max\left\{ \frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f)}, \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\sigma_h^{(p,q,t)L}(f)} \right\}$$

$$\leq \varlimsup_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \leq \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f)} .$$

Proof. From the definition of ${}_{v_n}\sigma_h^{(p,q,t)L}(f)$ and ${}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)$, we have for arbitrary positive ε and for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ that

$$\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right) \geq \left({}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)-\varepsilon\right)\times$$

$$\left[\log^{[q-1]}\left(\frac{1}{(1-r_1)\cdot\cdot(1-r_n)}\right)\cdot \exp^{[t+1]} L\left(\frac{1}{1-r_1},\cdot\cdot,\frac{1}{1-r_n}\right)\right]^{{}_{v_n}\rho_h^{(p,q,t)L}(f\circ g)}, \tag{2.17}$$

and

$$\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right) \leq \left({}_{v_n}\sigma_h^{(p,q,t)L}(f)+\varepsilon\right)\times$$

$$\left[\log^{[q-1]}\left(\frac{1}{(1-r_1)\cdot\cdot(1-r_n)}\right)\cdot \exp^{[t+1]} L\left(\frac{1}{1-r_1},\cdot\cdot,\frac{1}{1-r_n}\right)\right]^{{}_{v_n}\rho_h^{(p,q,t)L}(f)} \tag{2.18}$$

Now from (2.17), (2.18) and the condition ${}_{v_n}\rho_h^{(p,q,t)L}(f\circ g) = {}_{v_n}\rho_h^{(p,q,t)L}(f)$, it follows for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ that

$$\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \geqslant \frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)-\varepsilon}{{}_{v_n}\sigma_h^{(p,q,t)L}(f)+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary , we obtain from above that

$$\varliminf_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \geqslant \frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\sigma_h^{(p,q,t)L}(f)} . \tag{2.19}$$

Again for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity,

$$\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right) \leq \left({}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)+\varepsilon\right)\times$$

$$\left[\log^{[q-1]}\left(\frac{1}{(1-r_1)\cdot\cdot(1-r_n)}\right)\cdot \exp^{[t+1]} L\left(\frac{1}{1-r_1},\cdot\cdot,\frac{1}{1-r_n}\right)\right]^{{}_{v_n}\rho_h^{(p,q,t)L}(f\circ g)} \tag{2.20}$$

and for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$,

$$\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right) \geq \left({}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f) - \varepsilon\right) \times$$

$$\left[\log^{[q-1]}\left(\frac{1}{(1-r_1)\cdot\cdot(1-r_n)}\right) \cdot \exp^{[t+1]} L\left(\frac{1}{1-r_1}, \cdot\cdot, \frac{1}{1-r_n}\right)\right]^{{}_{v_n}\rho_h^{(p,q,t)L}(f)} . \tag{2.21}$$

Combining (2.20) and (2.21) and the condition ${}_{v_n}\rho_h^{(p,q,t)L}(f \circ g) = {}_{v_n}\rho_h^{(p,q,t)L}(f)$, we get for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \leq \frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f \circ g) + \varepsilon}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f) - \varepsilon} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\varliminf_{r_1, r_2, \cdots, r_n \to 1} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \leq \frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f \circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f)} . \tag{2.22}$$

Also for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity, it follows that

$$\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right) \leq \left({}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f) + \varepsilon\right) \times$$

$$\left[\log^{[q-1]}\left(\frac{1}{(1-r_1)\cdot\cdot(1-r_n)}\right) \cdot \exp^{[t+1]} L\left(\frac{1}{1-r_1}, \cdot\cdot, \frac{1}{1-r_n}\right)\right]^{{}_{v_n}\rho_h^{(p,q,t)L}(f)} . \tag{2.23}$$

Now from (2.17), (2.23) and the condition ${}_{v_n}\rho_h^{(p,q,t)L}(f \circ g) = {}_{v_n}\rho_h^{(p,q,t)L}(f)$, we obtain for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \geq \frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f \circ g) - \varepsilon}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f) + \varepsilon} .$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\varlimsup_{r_1, r_2, \cdots, r_n \to 1} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \geq \frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f \circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f)} . \tag{2.24}$$

Also for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$,

$$\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right) \leq \left({}_{v_n}\sigma_h^{(p,q,t)L}(f \circ g) + \varepsilon\right) \times$$

$$\left[\log^{[q-1]}\left(\frac{1}{(1-r_1)\cdot\cdot(1-r_n)}\right)\cdot\exp^{[t+1]}L\left(\frac{1}{1-r_1},\cdot\cdot,\frac{1}{1-r_n}\right)\right]^{{}_{v_n}\rho_h^{(p,q,t)L}(f\circ g)}. \tag{2.25}$$

In view of the condition ${}_{v_n}\rho_h^{(p,q,t)L}(f\circ g) = {}_{v_n}\rho_h^{(p,q,t)L}(f)$, it follows from (2.21) and (2.25) for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ that

$$\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \le \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)+\varepsilon}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\varlimsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \le \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f)}. \tag{2.26}$$

Again from the definition of ${}_{v_n}\sigma_h^{(p,q,t)L}(f)$ we get for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdot\cdot$ $\cdot$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right) \ge \left({}_{v_n}\sigma_h^{(p,q,t)L}(f)-\varepsilon\right)\times$$

$$\left[\log^{[q-1]}\left(\frac{1}{(1-r_1)\cdot\cdot(1-r_n)}\right)\cdot\exp^{[t+1]}L\left(\frac{1}{1-r_1},\cdot\cdot,\frac{1}{1-r_n}\right)\right]^{{}_{v_n}\rho_h^{(p,q,t)L}(f)} \tag{2.27}$$

Now from (2.25), (2.27) and the condition ${}_{v_n}\rho_h^{(p,q,t)L}(f\circ g) = {}_{v_n}\rho_h^{(p,q,t)L}(f)$, it follows for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \le \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)+\varepsilon}{{}_{v_n}\sigma_h^{(p,q,t)L}(f)-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\varliminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \le \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\sigma_h^{(p,q,t)L}(f)}. \tag{2.28}$$

Again for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right) \geqslant \left({}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)-\varepsilon\right)\times$$

$$\left[\log^{[q-1]}\left(\frac{1}{(1-r_1)\cdot\cdot(1-r_n)}\right)\cdot\exp^{[t+1]}L\left(\frac{1}{1-r_1},\cdot\cdot,\frac{1}{1-r_n}\right)\right]^{{}_{v_n}\rho_h^{(p,q,t)L}(f\circ g)}. \tag{2.29}$$

So combining (2.18) and (2.29) and in view of the condition ${}_{v_n}\rho_h^{(p,q,t)L}(f\circ g) = {}_{v_n}\rho_h^{(p,q,t)L}(f)$, we get for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \geqslant \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g) - \varepsilon}{{}_{v_n}\sigma_h^{(p,q,t)L}(f) + \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\overline{\lim_{r_1,r_2,\cdots,r_n\to 1}} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \geqslant \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\sigma_h^{(p,q,t)L}(f)}. \tag{2.30}$$

Thus the theorem follows from (2.19), (2.22), (2.24), (2.26), (2.28) and (2.30) □

The following theorem can be proved in the line of Theorem 5 and so its proof is omitted.

Theorem 6. *If $f(z_1, z_2, \cdot\cdot, z_n)$, $g(z_1, z_2, \cdot\cdot, z_n)$ be any two analytic functions in U and $h(z_1, z_2, \cdot\cdot, z_n)$ be an entire function of n-complex variables such that $0 < {}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g) \le {}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g) < \infty$, $0 < {}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(g) \le {}_{v_n}\sigma_h^{(p,q,t)L}(g) < \infty$ and ${}_{v_n}\rho_h^{(p,q,t)L}(f\circ g) = {}_{v_n}\rho_h^{(p,q,t)L}(g)$, where $p, q \in \mathbb{N}$ and $t \in \mathbb{N}\cup\{-1, 0\}$. Then*

$$\begin{aligned}
\frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\sigma_h^{(p,q,t)L}(g)} &\le \underline{\lim_{r_1,r_2,\cdots,r_n\to 1}} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_g\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \\
&\le \min\left\{\frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(g)}, \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\sigma_h^{(p,q,t)L}(g)}\right\} \\
&\le \max\left\{\frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(g)}, \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\sigma_h^{(p,q,t)L}(g)}\right\} \\
&\le \overline{\lim_{r_1,r_2,\cdots,r_n\to 1}} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1, r_2, \cdot\cdot, r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_g\left(r_1, r_2, \cdot\cdot, r_n\right)\right)} \le \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(g)}.
\end{aligned}$$

Now in the line of Theorem 5 and Theorem 6 respectively one can easily prove the following two theorems using the notion of relative $(p,q,t)\,L$ -th weak type and therefore their proofs are omitted.

Theorem 7. *If $f(z_1, z_2, \cdot\cdot, z_n)$, $g(z_1, z_2, \cdot\cdot, z_n)$ be any two analytic functions in U and $h(z_1, z_2, \cdot\cdot, z_n)$ be an entire function of n-complex variables such that $0 < {}_{v_n}\tau_h^{(p,q,t)L}(f\circ g) \le {}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g) < \infty$, $0 < {}_{v_n}\tau_h^{(p,q,t)L}(f) \le {}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f) < \infty$ and ${}_{v_n}\lambda_h^{(p,q,t)L}(f\circ g) = {}_{v_n}\lambda_h^{(p,q,t)L}(f)$, where $p, q \in \mathbb{N}$ and $t \in \mathbb{N}\cup\{-1, 0\}$. Then*

$$
\begin{aligned}
\frac{{}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f)} &\le \underline{\lim}_{r_1,r_2,\cdots,r_n\to 1}\frac{\log^{[p-1]}M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]}M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \\
&\le \min\left\{\frac{{}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\tau_h^{(p,q,t)L}(f)},\frac{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f)}\right\} \\
&\le \max\left\{\frac{{}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\tau_h^{(p,q,t)L}(f)},\frac{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f)}\right\} \\
&\le \overline{\lim}_{r_1,r_2,\cdots,r_n\to 1}\frac{\log^{[p-1]}M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]}M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \le \frac{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\tau_h^{(p,q,t)L}(f)}.
\end{aligned}
$$

Theorem 8. *If $f(z_1,z_2,\cdot\cdot,z_n)$, $g(z_1,z_2,\cdot\cdot,z_n)$ be any two analytic functions in U and $h(z_1,z_2,\cdot\cdot,z_n)$ be an entire function of n-complex variables such that $0<{}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)$ $\le {}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g)<\infty$, $0<{}_{v_n}\tau_h^{(p,q,t)L}(g)\le {}_{v_n}\overline{\tau}_h^{(p,q,t)L}(g)<\infty$ and ${}_{v_n}\lambda_h^{(p,q,t)L}(f\circ g)$ $={}_{v_n}\lambda_h^{(p,q,t)L}(g)$, where $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$. Then*

$$
\begin{aligned}
\frac{{}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(g)} &\le \underline{\lim}_{r_1,r_2,\cdots,r_n\to 1}\frac{\log^{[p-1]}M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]}M_h^{-1}\left(M_g\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \\
&\le \min\left\{\frac{{}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\tau_h^{(p,q,t)L}(g)},\frac{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(g)}\right\} \\
&\le \max\left\{\frac{{}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\tau_h^{(p,q,t)L}(g)},\frac{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(g)}\right\} \\
&\le \overline{\lim}_{r_1,r_2,\cdots,r_n\to 1}\frac{\log^{[p-1]}M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]}M_h^{-1}\left(M_g\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \le \frac{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\tau_h^{(p,q,t)L}(g)}.
\end{aligned}
$$

We may now state the following theorems without their proofs based on relative (p,q,t) L-th type and relative (p,q,t) L-th weak type:

Theorem 9. *If $f(z_1,z_2,\cdot\cdot,z_n)$, $g(z_1,z_2,\cdot\cdot,z_n)$ be any two analytic functions in U and $h(z_1,z_2,\cdot\cdot,z_n)$ be an entire function of n-complex variables such that $0<{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)$ $\le {}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)<\infty$, $0<{}_{v_n}\tau_h^{(p,q,t)L}(f)\le {}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f)<\infty$ and ${}_{v_n}\rho_h^{(p,q,t)L}(f\circ g)$ $={}_{v_n}\lambda_h^{(p,q,t)L}(f)$, where $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$. Then*

$$\frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f)} \le \varliminf_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}$$
$$\le \min\left\{\frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\tau_h^{(p,q,t)L}(f)}, \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f)}\right\}$$
$$\le \max\left\{\frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\tau_h^{(p,q,t)L}(f)}, \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f)}\right\}$$
$$\le \varlimsup_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \le \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\tau_h^{(p,q,t)L}(f)}.$$

Theorem 10. *If $f(z_1,z_2,\cdot\cdot,z_n)$, $g(z_1,z_2,\cdot\cdot,z_n)$ be any two analytic functions in U and $h(z_1,z_2,\cdot\cdot,z_n)$ be an entire function of n-complex variables such that $0 < {}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)$ $\le {}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g) < \infty$, $0 < {}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f) \le {}_{v_n}\sigma_h^{(p,q,t)L}(f) < \infty$ and ${}_{v_n}\lambda_h^{(p,q,t)L}(f\circ g)$ $= {}_{v_n}\rho_h^{(p,q,t)L}(f)$, where $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$. Then*

$$\frac{{}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\sigma_h^{(p,q,t)L}(f)} \le \varliminf_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}$$
$$\le \min\left\{\frac{{}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f)}, \frac{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\sigma_h^{(p,q,t)L}(f)}\right\}$$
$$\le \max\left\{\frac{{}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f)}, \frac{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\sigma_h^{(p,q,t)L}(f)}\right\}$$
$$\le \varlimsup_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_f\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \le \frac{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f)}.$$

Theorem 11. *If $f(z_1,z_2,\cdot\cdot,z_n)$, $g(z_1,z_2,\cdot\cdot,z_n)$ be any two analytic functions in U and $h(z_1,z_2,\cdot\cdot,z_n)$ be an entire function of n-complex variables such that $0 < {}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)$ $\le {}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g) < \infty$, $0 < {}_{v_n}\tau_h^{(p,q,t)L}(g) \le {}_{v_n}\overline{\tau}_h^{(p,q,t)L}(g) < \infty$ and ${}_{v_n}\rho_h^{(p,q,t)L}(f\circ g)$ $= {}_{v_n}\lambda_h^{(p,q,t)L}(g)$, where $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$. Then*

$$\frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(g)} \le \varliminf_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_g\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}$$
$$\le \min\left\{\frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\tau_h^{(p,q,t)L}(g)}, \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(g)}\right\}$$

$$\leq \max\left\{\frac{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\tau_h^{(p,q,t)L}(g)}, \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(g)}\right\}$$

$$\leq \overline{\lim_{r_1,r_2,\cdots,r_n\to 1}} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_g\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \leq \frac{{}_{v_n}\sigma_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\tau_h^{(p,q,t)L}(g)}.$$

Theorem 12. *If $f(z_1,z_2,\cdot\cdot,z_n)$, $g(z_1,z_2,\cdot\cdot,z_n)$ be any two analytic functions in U and $h(z_1,z_2,\cdot\cdot,z_n)$ be an entire function of n-complex variables such that $0 < {}_{v_n}\tau_h^{(p,q,t)L}(f\circ g) \leq {}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g) < \infty$, $0 < {}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(g) \leq {}_{v_n}\sigma_h^{(p,q,t)L}(g) < \infty$ and ${}_{v_n}\lambda_h^{(p,q,t)L}(f\circ g) = {}_{v_n}\rho_h^{(p,q,t)L}(g)$, where $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$. Then*

$$\frac{{}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\sigma_h^{(p,q,t)L}(g)} \leq \underline{\lim}_{r_1,r_2,\cdots,r_n\to 1} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_g\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}$$

$$\leq \min\left\{\frac{{}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(g)}, \frac{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\sigma_h^{(p,q,t)L}(g)}\right\}$$

$$\leq \max\left\{\frac{{}_{v_n}\tau_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(g)}, \frac{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\sigma_h^{(p,q,t)L}(g)}\right\}$$

$$\leq \overline{\lim_{r_1,r_2,\cdots,r_n\to 1}} \frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r_1,r_2,\cdot\cdot,r_n\right)\right)}{\log^{[p-1]} M_h^{-1}\left(M_g\left(r_1,r_2,\cdot\cdot,r_n\right)\right)} \leq \frac{{}_{v_n}\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{{}_{v_n}\overline{\sigma}_h^{(p,q,t)L}(g)}.$$

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The Aligarh Bulletin of Mathematics
Volume 37, Numbers 1-2 (2018), 21-38

ISSN: 0304-9787

On the isomorphism of two bases of exponentials in weighted Morrey type spaces

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(Received January 14, 2019)

Abstract

The paper considers a double system of exponentials with complex-valued coefficients. Under certain conditions on the coefficients it is proved that if this system forms a basis in a weighted Morrey-Lebesgue type space on the interval $[-\pi, \pi]$, then it is isomorphic to the classical system of exponentials in this space if the weight function satisfies certain conditions.

1 Introduction

Consider the double system of exponentials

$$\left\{A\left(t\right)e^{i\,nt};B\left(t\right)e^{-i\,nt}\right\}_{n\in Z_{+},k\in N}, \tag{1.1}$$

Keywords and phrases : Morrey space, system of exponential, basicity
2010 AMS Subject Classification : 33B10, 46E30, 54D70

with complex valued coefficients $A(t) = |A(t)| e^{i\alpha(t)}$; $B(t) = |B(t)| e^{i\beta(t)}$ on the interval $[-\pi, \pi]$, where $N-$ is a set of natural numbers, $Z_+ = \{0\} \bigcup N$. The system (1.1) is a generalization of the following binary system of cosines and sines

$$1 \bigcup \{\cos(nt + \gamma(t)); \sin(nt + \gamma(t))\}_{n \in Z_+}, \tag{1.2}$$

where, generally speaking, $\gamma : [-\pi, \pi] \to C-$ is a complex-valued function. A lot of work has been devoted to the study of basis properties (completeness, minimality, basicity) of systems of the form (1.1), (1.2) in spaces $L_p(-\pi, \pi)$, $1 \leq p < +\infty (L_\infty(-\pi, \pi) \equiv C[-\pi, \pi])$, starting with the classical results of Paley-Wiener [1] and N. Levinson [2] concerning the basis properties of perturbed systems of exponentials. The well-known "$\frac{1}{4}-$Kadets" theorem [3] also refers to this range of issue. The criterion for the basicity of a system of exponentials

$$\left\{e^{i\,(n + \alpha sign n)\,t}\right\}_{n \in Z}, \tag{1.3}$$

in $L_p(-\pi, \pi)$, $1 < p < +\infty$, was first found in the work of A. M. Sedletskii [4], where Z are integers. The same and other results were obtained by the method of boundary value problems by E. I.Moiseev [5]. Note that, single variants of these systems are the system of cosine

$$1 \bigcup \{\cos(n + \alpha) t\}_{n \in N}, \tag{1.4}$$

and the system of sine

$$\{\sin(n + \alpha) t\}_{n \in N}, \tag{1.5}$$

which arise when solving a series of equations of mixed type by the Fourier method (see, for example, works [6-10]). The basis properties of the systems (1.4), (1.5) in the spaces $L_p(0, \pi)$, $1 < p < +\infty$, are completely studied in the works of E. I. Moiseev [5, 11], when $\alpha \in R-$is a real parameter. These results were transferred to the case of complex parameter by G. G. Devdariani [12;13]. When $\gamma : [-\pi, \pi] \to C$ is a Holder function, the Riesz basicity of the system (1.2) in $L_2(-\pi, \pi)$, was studied in the work of A. N. Barmenkov [14]. One of the effective methods for studying the basis properties of systems of the form (1.1) - (1.5) is the method of boundary value problems of the theory of analytic functions, which originates from the work of A. V. Bitsadze [15]. This method was successfully used by the authors of works [5-14, 16-19]. B. T. Bilalov [16-18, 20] considered the most general case, namely, considering the systems of the form (1.1) and using the results concerning the basis properties of the system (1.1), he established a criterion for the basicity, completeness and minimality of a sine system of the form

$$\{\sin(nt + \gamma(t))\}_{n \in N}, \tag{1.6}$$

in $L_p(0, \pi)$, $1 < p < +\infty$, when $\gamma : [0, \pi] \to C-$ is a piecewise continuous function. Similar results concerning system (1.6) were obtained earlier in the paper [21].

The study of the basis properties of systems of the form (1.1) - (1.6) in various function spaces still continues. The weighted case of L_p space is considered in [22-24]. These problems are studied in Sobolev spaces in [25-27], the basicity of the system (1.3) is studied in the generalized Lebesgue spaces in [19;28]. It should be noted that, recently, interest in study of various problems of analysis in Morrey type spaces has greatly increased. There is a natural need to study the approximate properties of systems of the form (1.1) - (1.6) in Morrey type spaces. Some problems in the theory of approximation were studied in papers [29-31]. In the paper [32], the basis property of the classical system of exponentials is studied in Morrey-Lebesgue type spaces.

In this paper a double system of exponentials with complex-valued coefficients is considered. Under some conditions on the coefficients, it is proved that if this system forms a basis for a weighted Morrey-Lebesgue type space on the interval $[-\pi, \pi]$, if the weight function satisfies certain conditions then it is isomorphic to the classical system of exponentials in this space.

2 Necessary information

We need some information from the theory of Morrey type spaces. Let Γ be some rectifiable Jordan curve in the complex plane C. We denote by $|M|_\Gamma$ the linear Lebesgue measure of the set $M \subset \Gamma$. Everywhere in the future, the constants (may be different in different places) will be denoted by c.

By the Morrey-Lebesgue space $L^{p,\alpha}(\Gamma)$, $0 \le \alpha \le 1$, $p \ge 1$, we mean the normed space of all functions $f(\cdot)$ that are measurable on Γ with a finite norm $\|\cdot\|_{L^{p,\alpha}(\Gamma)}$:

$$\|f\|_{L^{p,\alpha}(\Gamma)} = \sup_B \left(\left| B \bigcap \Gamma \right|_\Gamma^{\alpha - 1} \int_{B \bigcap \Gamma} |f(\xi)|^p \, |d\xi| \right)^{1/p} < +\infty,$$

where B is an arbitrary ball with center on Γ.

$L^{p,\alpha}(\Gamma)$ is a Banach space and $L^{p,1}(\Gamma) = L_p(\Gamma)$, $L^{p,0}(\Gamma) = L_\infty(\Gamma)$. Inclusion $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ is true when $0 \le \alpha_1 \le \alpha_2 \le 1$. So that $L^{p,\alpha}(\Gamma) \subset L_1(\Gamma)$, $\forall \alpha \in [0, 1]$,$\forall p \ge 1$. The case $\Gamma = [-\pi, \pi]$ will be denoted by $L^{p,\alpha}(-\pi, \pi) = L^{p,\alpha}$. More detailed information on Morrey-type spaces can be obtained from [33-38].

Let $\omega = \{z \in C : |z| < 1\}$ be a unit ball in C and $\partial\omega = \gamma$ be a unit circle. We define the Morrey-Hardy space $H_+^{p,\alpha}$ of analytic functions $f(z)$ within ω with the norm $\|\cdot\|_{H_+^{p,\alpha}}$:

$$\|f\|_{H_+^{p,\alpha}} = \sup_{0<r<1} \|f(re^{n})\|_{L^{p,\alpha}}.$$

We denote by $\tilde{L}^{p,\alpha}$ the linear subspace $L^{p,\alpha}$ of functions whose shifts are continuous in $L^{p,\alpha}$, such that $\|f(\cdot+\delta)-f(\cdot)\|_{L^{p,\alpha}} \to 0$, when $\delta \to 0$. We take the closure of $\tilde{L}^{p,\alpha}$ in $L^{p,\alpha}$ and denote it by $M^{p,\alpha}$.

Let us consider the space $H_+^{p,\alpha}$. The subspace of $L^{p,\alpha}$ generated by the restrictions of functions from $H_+^{p,\alpha}$ to γ is denoted by $L_+^{p,\alpha}$. It follows immediately from the results of the preceding items that the spaces $H_+^{p,\alpha}$ and $L_+^{p,\alpha}$ are isomorphic and $f^+(\cdot) = (Jf)(\cdot)$, where $f \in H_+^{p,\alpha}$, f^+ are nontangential boundary values of f on γ, and J realizes the corresponding isomorphism. Let $M_+^{p,\alpha} = M^{p,\alpha} \bigcap L_+^{p,\alpha}$. It is clear that $M_+^{p,\alpha}$ is a subspace of $M^{p,\alpha}$ with respect to the norm $\|\cdot\|_{L^{p,\alpha}}$. Set $MH_+^{p,\alpha} = J^{-1}\left(M_+^{p\,\alpha}\right)$. It is a subspace of $H_+^{p,\alpha}$. Let $f \in H_+^{p,\alpha}$ and f^+ be its boundary values. It is obvious that the norm $\|f\|_{H_+^{p,\alpha}}$ can also be determined by the expression $\|f\|_{H_+^{p,\alpha}} = \|f^+\|_{L^{p,\alpha}}$. The case $\alpha = 1 \wedge p = 1$ will be denoted by H_1^+, i.e., $H_1^+ = H_+^{1,1}$.

Similarly to the classical case, we define the Morrey-Hardy class outside ω. So, let $\omega^- = C\backslash\bar{\omega}$ ($\bar{\omega} = \omega \bigcup \gamma$). We say that an analytic function f in ω^- has finite order k at infinity if the Laurent series of it in the neighborhood of an infinitely removable point has the form

$$f(z) = \sum_{n=-\infty}^{k} a_n z^n, k < +\infty, a_k \neq 0. \tag{2.1}$$

Thus, for $k > 0$ the function $f(z)$ has a pole of order k; for k = 0, it is bounded; and in the case $k < 0$ it has a zero order $(-k)$. Let $f(z) = f_0(z) + f_1(z)$, where $f_0(z)$ is the major, and $f_1(z)$ is a regular part of the expansion (2.1) of the function$f(z)$. Therefore, if $k < 0$ then $f_0(z) = 0$. For $k \geq 0$,$f_0(z)$ is a polynomial of degree k. We say that a function $f(z)$ belongs to the class ${}_mH_-^{p,\alpha}$if f has the order at infinity less than or equal to m, i.e., $k \leq m$ and $f_1\left(\frac{1}{z}\right) \in H_+^{p,\alpha}$. The case $\alpha = 1 \wedge p = 1$ will be denoted by ${}_mH_1^-$, i.e. ${}_mH_1^- ={}_m H_-^{1,1}$. The class ${}_mMH_-^{p,\alpha}$ is completely similarly defined to the case of $MH_+^{p,\alpha}$. In other words, ${}_mMH_-^{p,\alpha}$ is the subspace of functions in ${}_mH_-^{p,\alpha}$ whose shifts on the unit circle are continuous with respect to the norm $\|\cdot\|_{L^{p,\alpha}(\gamma)}$.

The weight case $L_\mu^{p,\alpha}(\Gamma)$ of the Morrey-Lebesgue space with weight function $\mu(\cdot)$ on Γ with the norm $\|\cdot\|_{L^{p,\alpha}(\Gamma)}$ is determined in a natural way

$$\|f\|_{L_\mu^{p,\alpha}(\Gamma)} = \|f\mu\|_{L^{p,\alpha}(\Gamma)}, f \in L_\mu^{p,\alpha}(\Gamma).$$

The inclusion $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ is valid for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Thus, $L^{p,\alpha}(\Gamma) \subset L_1(\Gamma)$, $\forall\alpha \in [0,1]$, $\forall p \geq 1$. The case $\Gamma = [-\pi,\pi]$ will be denoted by $L^{p,\alpha}(-\pi,\pi) = L^{p,\alpha}$.

We also use the following concepts. Let $\Gamma \subset C$ be same bounded rectifiable curve, and $t = t(\sigma)$, $0 \leq \sigma \leq 1$, its parametric representation with respect to the length of the arc σ

and l is the length of Γ. We set $d\mu(t) = d\sigma$, that is, $\mu(\cdot)$ is a linear measure on Γ. Let

$$\Gamma_t(r) = \{\tau \in \Gamma : |\tau - t| < r\}, \Gamma_{t(s)}(r) = \{\tau(\sigma) \in \Gamma : |\sigma - s| < r\}.$$

It is quite obvious that $\Gamma_{t(s)}(r) \subset \Gamma_t(r)$.

Definition 2.1. A curve X is called Carleson curve if $\exists c > 0$, such that

$$\sup_{t\in\Gamma} \mu(\Gamma_t(r)) \le cr, \forall r > 0.$$

It is said that the curve Γ satisfies the condition $arc-chord$ at the point $t_0 = t(s_0) \in \Gamma$ if there exists a constant $m > 0$, independent of t, such that

$$|s - s_0| \le m |t(s) - t(s_0)|, \forall t(s) \in \Gamma.$$

Γ satisfies the condition $arc - chord$ uniformly on Γ if

$$\exists m > 0 : |s - \sigma| \le m |t(s) - t(\sigma)|, \forall t(s), t(\sigma) \in \Gamma.$$

By S_Γ we denote the following singular integral operator

$$(S_\Gamma f)(\tau) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)\, d\zeta}{\zeta - \tau}, \ \tau \in \Gamma.$$

We shall essentially use the following theorem from N.Samko's paper [37].

Theorem 2.1 [37]. Suppose that the curve Γ satisfies the condition $arc - chord$ and the weight $\rho(\cdot)$ is defined by

$$\rho(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}; \ \{t_k\}_1^m \subset \Gamma, t_i \ne t_j, i \ne j.$$

A singular operator S_Γ is bounded in a weighted space $L_\rho^{p,\alpha}(\Gamma)$, $1 < p < +\infty$, $0 < \alpha \le 1$, if the inequalities

$$-\frac{\alpha}{p} < \alpha_k < -\frac{\alpha}{p} + 1, k = \overline{1, m}, \tag{2.2}$$

hold. Moreover, if Γ is smooth in some neighborhoods of the points t_k, $k = \overline{1, m}$, then the inequalities (2.2) are necessary for the boundedness of the operator S_Γ in $L_\rho^{p,\alpha}(\Gamma)$.

In the future, as Γ we will take the unit circle $\gamma = \partial\omega$. Consider the weighted space $L_\rho^{p,\alpha}(\gamma) =: L_\rho^{p,\alpha}$ with weight $\rho(\cdot)$. In a completely analogous manner to the weightless case, we define the space $M_\rho^{p,\alpha}$ with weight ρ. Thus, we denote by $\tilde{M}_\rho^{p,\alpha}$ the set of functions whose shifts are continuous in $L_{\rho}^{p,\,\alpha}$, that is

$$\|S_{\delta}f-f\|_{p,\,\alpha;\,\rho}=\|f\left(\cdot+\delta\right)-f\left(\cdot\right)\|_{p,\,\alpha;\,\rho}\to 0,\delta\to 0,$$

where S_{δ} is the shift operator: $\left(S_{\delta}f\right)(x)=f\left(x+\delta\right)$. It is easy to see that if $\rho\in L^{p,\,\alpha}$, then $C\left[-\pi,\,\pi\right]\subset M_{\rho}^{p,\,\alpha}$ holds. In fact, let $f\in C\left[-\pi,\,\pi\right]$. We have

$$\left|f\left(x+\delta\right)-f\left(x\right)\right|\leq\|f\left(\cdot+\delta\right)-f\left(\cdot\right)\|_{\infty}\to 0,\delta\to 0.$$

Consequently

$$\|f\left(\cdot+\delta\right)-f\left(\cdot\right)\|_{p,\,\alpha;\,\rho}=\|\left(f\left(\cdot+\delta\right)-f\left(\cdot\right)\right)\rho\left(\cdot\right)\|_{p,\,\alpha}$$

$$\leq\|f\left(\cdot+\delta\right)-f\left(\cdot\right)\|_{\infty}\|\rho\left(\cdot\right)\|_{p,\,\alpha}\to 0,\delta\to 0.$$

It follows that $f\in M_{\rho}^{p,\,\alpha}$.

Thus, the following is true

Lemma 2.1. Let $\nu\in L^{p,\lambda}\left(a,b\right)$. Then we have the inclusion $C[a,b]\subset M_{\nu}^{p,\lambda}\left(a,b\right)$.

The class of weights $\rho\left(\cdot\right)$, for which there is a continuous embedding $H_{+,\rho}^{p,\alpha}\subset H_{1}^{+}$, is denoted by $W^{p,\alpha}$, that is

$$W^{p,\alpha}=\left\{\rho\left(\cdot\right):H_{+,\rho}^{p,\alpha}\subset H_{1}^{+}\right\}.$$

Let $f\in H_{+,\rho}^{p,\alpha}$, $1<p<+\infty$, $0<\alpha<1$, and $\rho\in W^{p,\alpha}$. It is known that the relation

$$\lim_{r\to 1-0}\int_{-\pi}^{\pi}\left|f\left(re^{it}\right)-f^{+}\left(e^{it}\right)\right|\,dt=0, \tag{2.3}$$

holds, where $f^{+}\left(\cdot\right)$ is the nontangential boundary values of the function $f\in H_{1}^{+}$ on γ. It follows from (2.3) that there exists a sequence $\{r_n\}_{n\in N}:r_n\to 1-0\,,\;n\to\infty$, such that the sequence $\left\{f\left(r_ne^{it}\right)\right\}_{n\in N}$ converges almost everywhere to $f^{+}\left(e^{it}\right)$ on$(-\pi,\pi)$. Consequently, for an arbitrary interval $I\subset(-\pi,\pi)$ we have

$$\left|f\left(r_ne^{it}\right)\rho\left(t\right)\right|^{p}\to\left|f^{+}\left(e^{it}\right)\rho\left(t\right)\right|^{p},\;\text{ a. e. } t\in I.$$

On the other hand, we have

$$\frac{1}{|I|^{1-\alpha}}\int_{I}\left|f\left(r_ne^{it}\right)\rho\left(t\right)\right|^{p}dt\leq\sup_{I\subset(-\pi,\pi)}\left(\frac{1}{|I|^{1-\alpha}}\int_{I}\left|f\left(r_ne^{it}\right)\rho\left(t\right)\right|^{p}dt\right)\leq\|f\|_{H_{+,\rho}^{p,\alpha}}^{p}.$$

Applying the Fatou's theorem, we obtain

$$\int_I \left|f^+\left(e^{it}\right)\rho\left(t\right)\right|^p dt \le \sup_n \int_I \left|f\left(r_n e^{it}\right)\rho\left(t\right)\right|^p dt \le |I|^{1-\alpha}\left\|f\right\|^p_{H^{p,\alpha}_{+,\rho}}\,,\ \forall I\subset\left(-\pi,\pi\right).$$

It immediately follows that $f^+ \in L^{p,\alpha}_\rho$ and moreover $\left\|f^+\right\|_{p,\alpha;\rho} \le \left\|f\right\|_{H^{p,\alpha}_{+,\rho}}$ holds. Now, on the contrary, let $g\left(\cdot\right) \in L^{p,\alpha}_\rho$, $1<p<+\infty$, $0<\alpha<1$, and consider the Cauchy type integral

$$f\left(z\right)=\frac{1}{2\pi i}\int_\gamma \frac{g\left(\tau\right)}{\tau-z}d\tau\,,\ \ z\in\omega. \tag{2.4}$$

Applying the Sohocki-Plemel formula in (2.4), we obtain

$$f^+\left(\tau\right)=\frac{1}{2}g\left(\tau\right)+\left(Sg\right)\left(\tau\right)\,,\ \tau\in\gamma. \tag{2.5}$$

We denote by M the Hardy Littlewood operator

$$\left(Mg\right)\left(x\right)\equiv Mg\left(x\right)=\sup_{x\in I}\frac{1}{|I|}\int_I \left|g\left(e^{it}\right)\right|dt,$$

where $\sup$ is taken over all intervals $I\subset[-\pi,\pi]$. Further, we denote by $A_{p,\alpha}$ the class of weights $\rho\left(\cdot\right)$, for which the singular operator S and the maximal operator M behaves boundedly in the space $L^{p,\alpha}_\rho$, that is

$$A_{p,\alpha}\equiv\left\{\rho:\left\|S;M\right\|_{L^{p,\alpha}_\rho\to L^{p,\alpha}_\rho}<+\infty\right\}.$$

So, suppose that $\rho\in A_{p,\alpha}$. Then from (2.5) we obtain $f^+\left(\cdot\right)\in L^{p,\alpha}_\rho$. Let us show that $f\in H^{p,\alpha}_{+,\rho}$. Let

$$P_r\left(\varphi\right)=\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\varphi+r^2}\,,\ \ 0\le r<1,$$

be the Poisson kernel for the unit ball. It is known that (see, for example, [27, 41]) the function $f\in H_1^+$ is representable in terms of the Poisson-Lebesgue integral

$$f\left(re^{i\varphi}\right)=\int_{-\pi}^{\pi} f^+\left(e^{is}\right)P_r\left(\varphi-s\right)ds\,,\ \ z=re^{i\varphi}\in\omega.$$

Let $\Gamma_\alpha\left(\tau\right)$ be a nontangential angle with vertex $\tau\in\gamma$ and $\alpha\in\left(0,\pi\right)$. Then, as it is known (see [27])

$$\exists A_\alpha>0:\ \sup_{z\in\Gamma_\alpha\left(\tau\right)}\left|f\left(z\right)\right|\le A_\alpha Mf^+\left(\tau\right),\quad \text{a. e.}\quad \tau\in\gamma,$$

where $Mf^+\left(e^{it}\right)=Mg\left(t\right)\,,\ g\left(t\right)=f^+\left(e^{it}\right)$. From this we immediately obtain that for an arbitrary fixed $r \in (0,1)$, we have

$$\left|f\left(re^{it}\right)\right| \le AMf^{+}\left(e^{it}\right), \quad \text{a. e.} \quad t \in (-\pi,\pi),$$

where $A>0$ is an absolute constant. From here we immediately obtain

$$\|f_r(\cdot)\|_{p,\alpha;\rho} \le A\,\|Mg(\cdot)\|_{p,\alpha;\rho}\ ,\ \forall r \in (0,1),$$

where $f_r(t) = f\left(re^{it}\right)$. From the boundedness of the operator M in $L_{\rho}^{p,\alpha}$ implies

$$\sup_{0<r<1} \|f_r(\cdot)\|_{p,\alpha;\rho} \le C\left\|f^{+}\right\|_{p,\alpha;\rho},$$

where $C>0$ is some constant. Consequently, $f \in H_{+,\rho}^{p,\alpha}$. Thus, we have following **Theorem 2.2.** Let $f(\cdot) \in H_{+,\rho}^{p,\alpha}$, $1<p<+\infty$, $0<\alpha<1$, and $\rho \in W^{p,\alpha} \bigcap A_{p,\alpha}$. Then $f^{+}(\cdot) \in L_{\rho}^{p,\alpha}$ and the Cauchy formula

$$f(z) = \frac{1}{2\pi}\int_{\gamma} \frac{f^{+}(\tau)\,d\tau}{\tau - z}\ ,\ \ z \in \omega, \tag{2.6}$$

holds, where $f^{+}(\cdot)$ is the nontangential boundary value of $f(\cdot)$ on γ. Conversely, if $f^{+}(\cdot) \in L_{\rho}^{p,\alpha}$, then the function $f(\cdot)$, defined by a Cauchy type integral (2.6), belongs to the class $H_{+,\rho}^{p,\alpha}$, and we have

$$\|f\|_{p,\alpha;\rho} \le C\left\|f^{+}\right\|_{p,\alpha;\rho},$$

where $C>0$ is some constant. Let $f(\cdot)$ be the given function on $[a,b]$. In determining the Zorko type subspace we will assume that the function $f(\cdot)$ is continued to $[2a-b, 2b-a]$ with the following expression (and this function is also denoted by $f(\cdot)$

$$f(x) = \begin{cases} f(2a-x),\ x \in [2a-b, a), \\ f(2b-x),\ x \in (b, 2b-a]. \end{cases}$$

Since $M_{\nu}^{p,\lambda}(a,b)$ is a closed subspace of $\mathrm{L}_{\nu}^{p,\lambda}(a,b)$, it also contains the $\mathrm{L}_{\nu}^{p,\lambda}$ -closure of $C_0^{\infty}[a,b]$; in fact, $M_{\nu}^{p,\lambda}(a,b)$ is precisely that closure.

Proposition 2.1. *Let ν be given by the formula*

$$\nu(t) = \prod_{k=0}^{r} |t-t_k|^{\alpha_k}\ ,\ \ t \in [-\pi,\pi]\ ,\ \ t_i \ne t_j\ , i \ne j\ ,$$

and the conditions

$$\alpha_k \in \left[-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p}+1\right), \ k=\overline{0,r}\ , \tag{2.7}$$

hold. Then the set $C_0^\infty[-\pi,\pi]$ is dense in $M_\nu^{p,\lambda}(-\pi,\pi)$.

We need the following lemma.

Lemma 2.2 [Minkowski inequality for integrals in weighted Morrey spaces] Let $(X; X_\sigma; \mu)$ be a measurable space with an σ-additive measure $\mu(\cdot)$ on a set X, $\nu=\nu(t)$ a weight function, dy a linear Lebesgue measure on an interval (a,b) and $F(x,y)$ is $\mu\times dy$-measurable. If $1\le p<\infty$, then

$$\left\|\int_X F(x,y)d\mu(x)\right\|_{p,\lambda;\nu} \le \int_X \|F(x,y)\|_{p,\lambda;\nu}\, d\mu(x).$$

Proof. By using the Minkowski inequality for integrals in $L_p(a,b)$,

$$\left\|\int_X F(x,y)\nu(y)d\mu(x)\right\|_{L_p} \le \int_X \|F(x,y)\nu(y)\|_{L_p}\, d\mu(x),$$

we have

$$\left(\int_{B_r(x)}\left|\int_X F(x,y)\nu(y)d\mu(x)\right|^p dy\right)^{\frac{1}{p}} \le \int_X\left(\int_{B_r(x)}|F(x,y)\nu(y)|^p\, dy\right)^{\frac{1}{p}} d\mu(x),$$

where $B_r(x)$ is a ball with a radius $r>0$ and the center at $x\in X$. Then

$$\left(\frac{1}{r^\lambda}\int_{B_r(x)}\left|\int_X F(x,y)\nu(y)d\mu(x)\right|^p dy\right)^{\frac{1}{p}} \le \int_X\left(\frac{1}{r^\lambda}\int_{B_r(x)}|F(x,y)\nu(y)|^p\, dy\right)^{\frac{1}{p}} d\mu(x).$$

The required result follows by taking the supremum over all $x\in(a,b)$ and $r>0$ in the last inequality.

It is now easy to provide the proof of Proposition 2.1.

Proof of Proposition 2.1. Let $f\in M_\nu^{p,\lambda}(-\pi,\pi)$, and $\varepsilon>0$, be a sufficiently small number. Consider the function

$$w_\varepsilon(t)=\begin{cases} c_\varepsilon e^{\left(\frac{-\varepsilon^2}{\varepsilon^2-t^2}\right)}, & |t|<\varepsilon,\\ 0, & |t|\ge\varepsilon,\end{cases}$$

where c_ε is chosen such that $\int_{-\infty}^{\infty} w_\varepsilon(t)dt=1$. Define the function $f_\varepsilon(t)$ as

$$f_\varepsilon(t)=\int_{-\infty}^{\infty} w_\varepsilon(s)f(t-s)ds.$$

As $\varepsilon>0$ is sufficiently small, this definition is correct. In fact, it is enough to prove that$f \in L_1(-\pi,\pi)$. It follows from $f \in M_{\nu}^{p,\lambda}(-\pi,\pi)$ that $(f\nu) \in L_{p,\lambda}(-\pi,\pi)$. Suppose that conditions (2.7) are satisfied. Then, it is not difficult to establish that $\nu^{-1} \in \left(L^{p,\lambda}(-\pi,\pi)\right)'$. Since $(f\nu) \in L_{p,\lambda}(-\pi,\pi)$, it is clear that $f = (f\nu)\,\nu^{-1} \in L_1(-\pi,\pi)$.

It is clear that $f_{\varepsilon}(t)$ is infinitely differentiable function on $[-\pi,\pi]$, and

$$\|f_{\varepsilon} - f\|_{p,\lambda;\nu} = \left\|\int_{-\infty}^{\infty} w_{\varepsilon}(s) f(t-s)ds - f(t)\right\|_{p,\lambda;\nu} =$$
$$= \left\|\int_{-\infty}^{\infty} w_{\varepsilon}(s)\left[f(t-s) - f(t)\right]ds\right\|_{p,\lambda;\nu}.$$

Applying Lemma 2.2, we get

$$\|f_{\varepsilon} - f\|_{p,\lambda;\nu} \le \int_{-\infty}^{\infty} \|w_{\varepsilon}(s)\left[f(.-s) - f(.)\right]\|_{p,\lambda;\nu} ds \le$$
$$\le \sup_{|s|<\varepsilon} \|[f(.-s) - f(.)]\|_{p,\lambda;\nu} \int_{-\varepsilon}^{\varepsilon} w_{\varepsilon}(s)ds$$
$$= \sup_{|s|<\varepsilon} \|[f(.-s) - f(.)]\|_{p,\lambda;\nu} \to 0 \ \text{ as}\varepsilon \to 0.$$

This completes the proof.

By similar way we can define $M_{\nu}^{p,\lambda}(0,\pi)$ and prove the following proposition.

Proposition 2.2. Let ν be given as

$$\nu(t) = \prod_{k=0}^{r} |t - t_k|^{\alpha_k}, \quad t \in [0,\pi], \quad t_i \ne t_j, i \ne j, \tag{2.8}$$

where $t_0 = 0, t_r = \pi$, and t_k are arbitrary finite points in the interval $(0,\pi)$ for all $k = 1,2,...,r-1$ and $\alpha_k \in \mathrm{R}$ for all $k = 0,1,...,r$ and the conditions (2.7) be satisfied. Then the set $C^{\infty}[0,\pi]$, of all infinitely differentiable functions with compact support in $(0,\pi)$, is dense in $M_{\nu}^{p,\lambda}(0,\pi)$.

Concerning the basicity of the system of exponents in $M_{\nu}^{p,\lambda}(-\pi,\pi)$, we have the following theorem.

Theorem 2.3. Let ν be given as in (2.9).

(I) The system $\left\{e^{int}\right\}_{n\in \mathrm{Z}}$ is minimal in $\mathrm{L}_{\nu}^{p,\lambda}(-\pi,\pi)$ if $\alpha_k \in \left[\frac{\lambda-1}{p}, \frac{1-\lambda}{q} + \lambda\right)$ for all $k = 0,1,...,r$.

(II) The system $\left\{e^{int}\right\}_{n\in \mathrm{Z}}$ is complete in $M_{\nu}^{p,\lambda}(-\pi,\pi)$ if the following conditions α_0 ; $\alpha_r \in \left(-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p} + 1\right)$, $\alpha_k \in \left[-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p} + 1\right)$, $k = \overline{1, r-1}$ are satisfied.

(III) The system $\left\{e^{int}\right\}_{n\in \mathrm{Z}}$ forms a basis for $M_{\nu}^{p,\lambda}(-\pi,\pi)$ if and only if the following conditions $\alpha_k \in \left[-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p} + 1\right)$, $k = \overline{0, r}$ are satisfied.

We need the Sokhotskii-Plemel formula for the boundary values of a Cauchy type integral

$$\Phi\left(z\right)=\frac{1}{2\pi i}\int_{\gamma}\frac{f\left(\tau\right)d\tau}{\tau-z},$$

where $f\left(e^{it}\right)\in L_1\left(-\pi,\,\pi\right)$. Then the boundary values $\Phi^{\pm}\left(\tau\right)$, $\tau\in\gamma$, satisfy the following Sokhotsky-Plemelj expression

$$\Phi^{\pm}\left(\tau\right)=\pm\frac{1}{2}f\left(\tau\right)+\left(Sf\right)\left(\tau\right),\quad \text{a.e. on}\quad \tau\in\gamma, \tag{2.9}$$

where $S\left(\cdot\right)$ is the singular integral

$$\left(Sf\right)\left(\tau\right)=\frac{1}{2\pi i}\int_{\gamma}\frac{f\left(\xi\right)d\xi}{\xi-\tau},\tau\in\gamma.$$

We show that if all the conditions of Theorem 2.2 are satisfied, then the following direct decomposition

$$L_{\rho}^{p,\alpha}=H_{+,\rho}^{p,\,\alpha}\dot{+}\,{}_{-1}H_{-,\rho}^{p,\,\alpha},1<p<+\infty,0<\alpha<1, \tag{2.10}$$

holds, where $\rho\in W^{p,\,\alpha}\bigcap A_{p,\alpha}$ is some weight function. In fact, let $f\in L_{\rho}^{p,\alpha}$. Then it is clear that $f\in L_{p,\rho}$. We consider the Cauchy integral (i.e., the Cauchy formula)

$$\Phi^{+}\left(z\right)=\frac{1}{2\pi\,i}\int_{\gamma}\frac{\Phi^{+}\left(\tau\right)}{\tau-z}d\tau\,,\ |z|<1.$$

Then, by Theorem 2.2, we have $\Phi^{+}\left(z\right)\in H_{+,\rho}^{p,\,\alpha}$ for $|z|<1$. In a similar way, we have that the Cauchy integral

$$\Phi^{-}\left(z\right)=\frac{1}{2\pi\,i}\int_{\gamma}\frac{\Phi^{-}\left(\tau\right)}{\tau-z}d\tau\,,\ |z|>1,$$

belongs to the class ${}_{-1}H_{-,\rho}^{p,\,\alpha}$. It follows from the Sokhotski-Plemelj formula (2.10) that

$$f\left(\tau\right)=\Phi^{+}\left(\tau\right)-\Phi^{-}\left(\tau\right),\quad \text{almost everywhere on}\ \ \tau\in\gamma. \tag{2.11}$$

Thus, the expansion (2.4) takes place. It is easy to see that $H_{+,\rho}^{p,\,\alpha}\subset H_1^{+}\wedge{}_{-1}H_{-,\rho}^{p,\,\alpha}\subset{}_{-1}H_1^{-}$ holds. Then it follows from $H_1^{+}\bigcap{}_{-1}H_1^{-}=\{0\}$ that the decomposition (1.1) is unique.

The subspace of $L_{\rho}^{p,\,\alpha}$ generated by the restrictions of functions from ${}_{-1}H_{-,\rho}^{p,\,\alpha}$ to γ is denoted by ${}_{-1}L_{-,\rho}^{p,\,\alpha}$. Hence, we have $L_{+,\rho}^{p,\,\alpha}\bigcap{}_{-1}L_{-,\rho}^{p,\,\alpha}=\{0\}$. Then, from (2.9) follows immediately a direct expansion

$$L_{\rho}^{p,\,\alpha}=L_{+,\rho}^{p,\,\alpha}\dot{+}\,{}_{-1}L_{-,\rho}^{p,\,\alpha}. \tag{2.12}$$

Identifying $H^{p,\alpha}_{+,\rho} \leftrightarrow L^{p,\alpha}_{+,\rho}$ and ${}_{-1}L^{p,\alpha}_{-,\rho} \leftrightarrow {}_{-1}H^{p,\alpha}_{-,\rho}$, we obtain the expansion (2.10).
We similarly establish that, there is also a direct expansion

$$M^{p,\alpha}_{\rho} = MH^{p,\alpha}_{+,\rho} \dot{+} {}_{-1}MH^{p,\alpha}_{-,\rho}. \tag{2.13}$$

Let $P^{\pm}$ be the projections

$$P^{+} : M^{p,\alpha}_{\rho} \to M^{p,\alpha}_{+,\rho} \wedge P^{-} : M^{p,\alpha}_{\rho} \to {}_{-1}M^{p,\alpha}_{-,\rho},$$

generated by the decomposition (2.13). We denote by $T^{\pm} : M^{p,\alpha}_{\rho} \to M^{p,\alpha}_{\rho}$ the multiplication operators defined by the expressions

$$T^{+}f = Af \wedge T^{-}f = Bf, \forall f \in M^{p,\alpha}_{\rho}.$$

Suppose that the condition $A^{\pm 1}$; $B^{\pm 1} \in L_{\infty}(-\pi, \pi)$ holds. Assume that the system (2.3) forms a basis in $M^{p,\alpha}_{\rho}$. We take $\forall g \in M^{p,\alpha}_{\rho}$, and expand it on this basis

$$g(t) = A(t) \sum_{n=0}^{\infty} g_n e^{int} + B(t) \sum_{n=1}^{\infty} g_{-n} e^{-int}.$$

Since $A^{\pm 1} \in L_{\infty} \wedge B^{\pm 1} \in L_{\infty}$, it follows that the series $f^{+}(t) = \sum_{n=0}^{\infty} g_n e^{int}$ and $f^{-}(t) = \sum_{n=0}^{\infty} g_{-n} e^{-int}$ represent some functions of $M^{p,\alpha}_{\rho}$. We set

$$f(t) = \sum_{n=-\infty}^{+\infty} g_n e^{int}, t \in [-\pi, \pi].$$

It is clear that $f \in M^{p,\alpha}_{\rho}$. Let us show that the inclusions

$$f^{+} \in M^{p,\alpha}_{+,\rho} \wedge f^{-} \in {}_{-1}M^{p,\alpha}_{-,\rho}$$

occur. In fact, we have

$$\int_{\gamma} f^{+}(\arg \xi) \xi^{n} d\xi = i \int_{-\pi}^{\pi} f^{+}(t) e^{i(n+1)t} dt = i \sum_{n=0}^{\infty} g_k \int_{-\pi}^{\pi} e^{i(k+n+1)t} dt = 0, \forall n \in Z_{+}.$$

Then it follows from Privalov theorem [40] that $f^{+}(t)$ a boundary value of $F^{+} \in H^{+}_{1}$, and

$$F^{+}(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\arg \xi)}{\xi - z} d\xi, |z| < 1. \tag{2.14}$$

We have

$$F^{+}(z)=\frac{1}{2\pi i}\int_{\gamma}\sum_{n=0}^{\infty}g_n\frac{e^{int}}{e^{it}-z}de^{it}=\frac{1}{2\pi i}\sum_{n=0}^{\infty}g_n\int_{\gamma}\frac{\xi^n d\xi}{\xi-z}=\sum_{n=0}^{\infty}g_n z^n, |z|<1.$$

Since $f\in M_{\rho}^{p,\alpha}$, it follows from (2.14) that $F\in MH_{+,\rho}^{p,\alpha}$. Similarly, we can prove that $F^{-}\in {}_{-1}MH_{-,\rho}^{p,\alpha}$, where

$$F^{-}(z)=\sum_{n=1}^{\infty}g_{-n}z^{-n}, |z|>1.$$

Consider the operator $T=T^{+}P^{+}+T^{-}P^{-}$. We have

$$Tf=T^{+}P^{+}f+T^{-}P^{-}f=T^{+}f^{+}+T^{-}f^{-}=A(\cdot)f^{+}+B(\cdot)f^{-}=g.$$

Consequently, the equation

$$Tf=g, g\in M_{\rho}^{p,\alpha}, \tag{2.15}$$

has a solution for $\forall g\in M_{\rho}^{p,\alpha}$ in $M_{\rho}^{p,\alpha}$, that is, $R_T=M_{\rho}^{p,\alpha}$, where R_T is the range of the operator T. Let $f\in KerT$. We expand f with respect to the basis $\left\{e^{int}\right\}_{n\in Z}$:

$$f(t)=\sum_{n=-\infty}^{+\infty}f_n e^{int}.$$

We have

$$0=Tf=A(t)\sum_{n=0}^{\infty}f_n e^{int}+B(t)\sum_{n=1}^{\infty}f_{-n}e^{-int}.$$

Since the system (2.3) forms a basis in $M_{\rho}^{p,\alpha}$, it follows that $f_n=0, \forall n\in Z$, i.e., $KerT=\{0\}$. Since $T\in L\left(M_{\rho}^{p,\alpha}\right)$, it follows from the Banach theorem that $T^{-1}\in L\left(M_{\rho}^{p,\alpha}\right)$, which in turn means the correct solvability of equation (2.15). Now, on the contrary, let equation (2.15) be correctly solvable in $M_{\rho}^{p,\alpha}$. We take $\forall g\in M_{\rho}^{p,\alpha}$, and let $f=T^{-1}g$. We expand f with respect in the basis $\left\{e^{int}\right\}_{n\in Z}$ in $M_{\rho}^{p,\alpha}$:

$$f(t)=\sum_{n=-\infty}^{+\infty}f_n e^{int}, t\in[-\pi, \pi].$$

We have

$$P^{+}f=\sum_{n=0}^{\infty}f_n e^{int}; P^{-}f=\sum_{n=1}^{\infty}f_{-n}e^{-int},$$

and hence

$$Tf = A(t)\sum_{n=0}^{\infty} f_n e^{int} + B(t)\sum_{n=1}^{\infty} f_{-n} e^{-int} = g(t),$$

i.e., an arbitrary element of $M_\rho^{p,\alpha}$ decomposes along the system (1.1) in $M_\rho^{p,\alpha}$. We show that such a decomposition is unique. Let

$$A(t)\sum_{n=0}^{\infty} f_n e^{int} + B(t)\sum_{n=1}^{\infty} f_{-n} e^{-int} = 0.$$

We set

$$f(t) = \sum_{n=-\infty}^{+\infty} f_n e^{int}.$$

It's clear that$f \in M_\rho^{p,\alpha}$. We have

$$Tf = A(t)\sum_{n=0}^{\infty} f_n e^{int} + B(t)\sum_{n=0}^{\infty} f_{-n} e^{-int} = 0 \Rightarrow f = T^{-1}0 = 0 \Rightarrow f_n = 0, \forall n \in Z.$$

Thus, the following theorem is proved:

Theorem 2.4. Let $A^{\pm 1}$;$B^{\pm 1} \in L_\infty(-\pi, \pi)$,and $\rho \in W^{p,\alpha} \bigcap A_{p,\alpha}$. The system (1.1) forms a basis in $M_\rho^{p,\alpha}$ if and only if equation (2.15) is correctly solvable in $M_\rho^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \leq 1$.

Now let's prove the following theorem.

Theorem 2.5. Let $A^{\pm 1}$;$B^{\pm 1} \in L_\infty(-\pi, \pi)$, and $\rho \in AM_p$. If the system (2.3) forms a basis in $M_\rho^{p,\alpha}$, then it is isomorphic to the classical system of exponentials $\left\{e^{int}\right\}_{n\in Z}$ in $M_\rho^{p,\alpha}$, and the isomorphism is given by the operator T_0:

$$(T_0 f)(t) = A(t)\sum_{n=0}^{\infty} \left(f;\, e^{inx}\right) e^{int} + B(t)\sum_{n=1}^{\infty} \left(f;\, e^{-inx}\right) e^{-int}, \qquad (2.16)$$

where

$$(f;\, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\overline{g(t)}dt.$$

Proof. Let the system (2.3) form a basis in $M_\rho^{p,\alpha}$. We take $\forall f \in M_\rho^{p,\alpha}$. Since the system $\left\{e^{int}\right\}_{n\in Z}$ forms a basis in $M_\rho^{p,\alpha}$, it is clear that the series

$$f^+(t) = \sum_{n=0}^{\infty} \left(f;\, e^{inx}\right) e^{int}, \quad f^-(t) = \sum_{n=1}^{\infty} \left(f;\, e^{-inx}\right) e^{-int},$$

converge in $M_\rho^{p,\alpha}$, and moreover,

$$\left\|f^{\pm}\right\|_{p,\,\alpha} \leq c\,\|f\|_{p,\,\alpha}$$

holds. Then it follows immediately from the expression (2.16) of the operator T_0 that $T_0 \in L\left(M_\rho^{p,\alpha}\right)$. We show that $KerT_0 = \{0\}$. Let $f \in KerT_0$, i.e.,

$$T_0 f = A\left(t\right) \sum_{n=0}^{\infty} \left(f;\, e^{inx}\right) e^{int} + B\left(t\right) \sum_{n=1}^{\infty} \left(f;\, e^{-inx}\right) e^{-int} = 0.$$

Since, system (2.3) forms a basis in $M_\rho^{p,\alpha}$, it follows that

$$\left(f;\, e^{inx}\right) = 0, \forall n \in Z \Rightarrow f = 0.$$

Since, the system $\left\{e^{inx}\right\}_{n \in Z}$ forms a basis in $M_\rho^{p,\alpha}$. Therefore, $KerT_0 = \{0\}$. And now, let's show that $R_{T_0} = M_\rho^{p,\alpha}$. Let $g \in M_\rho^{p,\alpha}$ be an arbitrary element. By Theorem 2.1 $\exists f \in M_\rho^{p,\alpha}$: $Tf = g$. On the other hand, it is not difficult to see that $T_0 = T$, and as a result $R_T = M_\rho^{p,\alpha}$. Then it follows from the Banach theorem that T_0 is an automorphism in $M_\rho^{p,\alpha}$. It's clear that $T_0\left[e^{inx}\right] = A\left(t\right) e^{int}$, $\forall n \in Z_+$, & $T_0\left[e^{-inx}\right] = B\left(t\right) e^{-int}$, $\forall n \in N$. The theorem is proved.

This theorem immediately implies following

Corollary 2.1. If the perturbed system of exponents

$$\left\{e^{i\left(n + \alpha sign n\right) t}\right\}_{n \in Z},$$

forms a basis in $M_\rho^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \leq 1$, then it is isomorphic to its classical exponents system $\left\{e^{int}\right\}_{n \in Z}$.

This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan - Grant No EIF/GAM-4- BGM-GIN-2017-3(29).

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The Aligarh Bulletin of Mathematics
Volume 37, Numbers 1-2 (2018), 39-45

ISSN: 0304-9787

Visible points of convex sets in metric spaces

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(Received January 15, 2019)

Abstract

The concept of visible points of a convex set was introduced and discussed in normed linear spaces by Frank Deutsch, Hein Hundal and Ludmil Zikatanov [Visible Points in Convex Sets and Best Approximation, Computational and Analytical Mathematics, Springer (2013), 349-364]. Extending this concept to metric spaces, we study some basic properties of such points, besides giving some characterizations of visible sets. We also study the connection between visible points and best approximation in such spaces. Moreover, we show that in linear metric spaces, those closed convex sets C for which the set of visible points to each point not in C is the whole set C are precisely the affine sets.

1 Introduction

Let C be a closed convex subset of a real normed linear space X and $x \in X$. An element $v \in C$ is said to be visible to x with respect to C if $[x, v] \cap C = \{v\}$ or, equivalently,

Keywords and phrases : Linear metric space, convex space, strongly convex space, M-space, best approximation, visible point
2010 AMS Subject Classification : 41A65, 41A50, 52A27

$[x, v[\cap C = \phi$. The notion of visible points was introduced and discussed in normed linear spaces by Deutsch et al. [2]. As remarked in [2], this concept is useful in the study of best approximation, and it also seems to have potential value in the study of robotics.

Geometrically, one can regard the set $V_C(x) = \{v \in C : [x, v] \cap C = \{v\}\} = \{v \in C : [x, v[\cap C = \phi\}$ as the "light" that would be cast on the set C if there were a light source at the point x emanating in all directions. Alternatively, one can regard the set C as an "obstacle" in X, a robot is located at a point $x \in X$, and the direction determined by the interval $[x, v]$, where $v \in V_C(x)$, as directions to be avoided by the robot so as not to collide with the obstacle C.

In this paper, we extend the notion of visible points to metric spaces, study some basic properties of such points, give some characterizations of visible sets and study the connection between visible points and best approximation in such spaces. We also show that in linear metric spaces, those closed convex sets C for which the set of visible points to each point not in C is the whole set C are precisely the affine sets.

2 Notations and Definitions

To start with, we recall a few definitions.

Definition 1. *[4] Let (X, d) be a metric space and $x, y, z \in X$. We say that z is* ***between*** *x and y if $d(x, z) + d(z, y) = d(x, y)$. For any two points $x, y \in X$, the set*

$$\{z \in X : d(x, z) + d(z, y) = d(x, y)\}$$

is called a ***metric segment*** *and is denoted by $[x, y]$.*

Definition 2. *[1] A metric space (X, d) is said to be* ***convex*** *if for every x, y in X and for every t, $0 \le t \le 1$ there exists at least one point z such that $d(x, z) = (1 - t)\, d(x, y)$ and $d(z, y) = t\, d(x, y)$.*

The space X is said to be ***strongly convex*** *[1] if such a z exists and is unique for each pair x, y of X. Thus in strongly convex metric spaces, each t, $0 \le t \le 1$, determines a unique point of the segment $[x, y]$.*

It was proved by Menger [4], that in a complete convex metric space (X, d), each two points $x, y \in X$ are joined by a metric segment [x,y] i.e. by a subset of X isometric with the real line interval of length $d(x, y)$.

Definition 3. *[3] A convex metric space (X, d) is called an M-**space** if for every two points $x, y \in X$ with $d(x, y) = \lambda$, and for every $r \in [0, \lambda]$, there exists a unique $z_r \in X$ such that*

$$B[x, r] \cap B[y, \lambda - r] = \{z_r\},$$

where $B[x, r] = \{y \in X : d(x, y) \le r\}$.

Clearly, every normed linear space is an M-space.

It is known (see[8]) that a metric space (X, d) is an M-space if and only if X is strongly convex.

Definition 4. *In a metric space (X, d), we define*

$$\begin{aligned}
[x, y] &= \{z \in X : d(x, y) = d(x, z) + d(z, y)\}, \\
[x, y[&= \{z \in X : d(x, y) = d(x, z) + d(z, y), z \neq y\}, \\
]x, y] &= \{z \in X : d(x, y) = d(x, z) + d(z, y), z \neq x\}, \\
]x, y[&= \{z \in X : d(x, y) = d(x, z) + d(z, y), z \neq x, z \neq y\}.
\end{aligned}$$

Definition 5. *(see [5]) A subset C of a metric space (X, d) is said to be **convex** if for every $x, y \in C$, any point between x and y is also in C i.e. for each $x, y \in C$, the metric segment $[x, y]$ lies in C.*

Definition 6. *[2] Let C be a closed convex subset of a metric space (X, d) and $x \in X$. A point $v \in C$ is said to be **visible** to x with respect to C if $[x, v] \cap C = \{v\}$ or, equivalently, $[x, v[\cap C = \phi$. The set of all visible points to x with respect to C is denoted by $V_C(x)$. Thus $V_C(x) = \{v \in C : [x, v] \cap C = \{v\}\} = \{v \in C : [x, v[\cap C = \phi\}$.*

Definition 7. *[9] A point $y_o \in C$ is said to be a **best approximation** to x in C if $d(x, y_o) = d(x, C) \equiv \inf\{d(x, c) : c \in C\}$. The set of all best approximations to x in C is denoted by $P_C(x)$ i.e.; $P_C(x) = \{y \in C : d(x, y) = d(x, C)\}$*

*The set C is said to be **proximinal** if $P_C(x)$ is non-empty for each $x \in X$. For examples of proximinal and non proximinal sets, we refer to Singer [9].*

3 Visible Points in Metric Spaces

In this section, we give some auxiliary results

Lemma 1. *Let C be a closed convex set in a metric space (X, d), if $x \in C$ then $V_C(x) = \{x\}$.*

Proof. Suppose $x \in C$, then $[x, x] = \{x\}$ and $[x, x] \cap C = \{x\} \cap C = \{x\}$. Therefore $x \in V_C(x)$ i.e. $\{x\} \subseteq V_C(x)$.

Conversely, let $v \in V_C(x)$ be any element and $x \in C$. To prove $v = x$. On contrary, let $v \neq x$. Now $v \in V_C(x)$ implies $[x, v] \cap C = \{v\}$. But $x \in C$, therefore $x \in [x, v] \cap C$. So $x = v$ and hence $V_C(x) = \{x\}$. □

It may be remarked that an analogous result is true for $P_C(x)$ i.e.; $P_C(x) = \{x\}$ if $x \in C$. Whereas the set $P_C(x)$ may be empty if $x \notin C$ (see [9]), the following theorem shows that the set $V_C(x)$ is always non-empty.

Theorem 1. *Let C be a closed convex set in an M-space (X, d), then*

(i) $V_C(x) \neq \phi$ *for each* $x \in X$ *and*

(ii) $V_C(x) \subset bdC$ *for each* $x \in X \diagdown C$.

Proof. (i) Let $x \in X$. By Lemma 1 we may assume that $x \notin C$ as if $x \in C$ then $V_C(x) = \{x\} \neq \phi$. Fix any $y \in C$. Then $[x, y]$ contains some points in C (e.g. y) and some points not in C (e.g. x). Let $t_o = \sup\{t, 0 < t < 1, z \in [x, y], d(x, z) = (1 - t)d(x, y)$ and $d(z, y) = td(x, y), z \in C\}$. Let $z_o \in [x, y]$ be such that $d(x, z_o) = (1 - t_o)d(x, y)$ and $d(z_o, y) = t_o d(x, y)$. Such a $z_o \in C$ as C is closed and $[x, z_o] \cap C = \{z_o\}$ i.e. $z_o \in V_C(x)$ and hence $V_C(x) \neq \phi$.

(ii) Fix any $x \in X \diagdown C$. To show that $v \in bdC$ for each $v \in V_C(x)$. If it is not so, then there exists some $v \in V_C(x)$ such that $v \in C \diagdown bdC$. Hence v is in the interior of C i.e. there exists an open ball $B(v, \epsilon) \subset C$. Let $v_o \in B(v, \epsilon)$ be such that $v_o \in [x, v]$. Then clearly $[v_o, v]$ is a subinterval of $[x, v]$ which lies inside C. Hence $[x, v] \cap C \neq \{v\}$, which contradicts the fact that $v \in V_C(x)$.

□

It may be remarked that result analogous to (ii) is true for $P_C(x)$ even if C is not convex (see [6]) but it is not true in metric spaces (see Singer [9])

The following theorem gives a characterization of visible points:

Theorem 2. *Let C be a closed convex set in an M-space (X, d), $x \in X \diagdown C$ and $v \in C$. Then the following are equivalent.*

(i) *v is visible to x with respect to C.*

(ii) $z \notin C$ *for* $t, 0 \leq t < 1, z \in [x, v], d(x, z) = (1 - t)d(x, v)$ *and* $d(z, v) = td(x, v)$.

(iii) $\max\{t, t \in [0, 1] : z \in C, z \in [x, v], d(x, z) = (1-t)d(x, v), d(z, v) = td(x, v)\} = 1$

Proof. (i) $\Rightarrow$ (ii). If (i) holds then $[x, v[\cap C = \phi$, then clearly, no $z \in [x, v]$, satisfying $d(x, z) = (1 - t)d(x, v)$ and $d(z, v) = td(x, y)$, $0 \leq t < 1$, belongs to C i.e. (ii) is true.

(ii) $\Rightarrow$ (iii). Since $v \in C$, (iii) is an obvious consequence of (ii).

(iii) $\Rightarrow$ (i). If (iii) is true then $[x, v[\cap C = \phi$ i.e. $v \in V_C(x)$. □

It is known (see [6]) that if C is a convex set in a convex metric space (X, d), then $P_C(x)$ is convex. It is also known (see [9]) that in any metric space, the set $P_C(x)$ is closed if C is closed. However, analogous results are not true for $V_C(x)$ even in normed linear spaces (see [2]).

The following theorem connects visible points and points of best approximation.

Theorem 3. *Let (X, d) be a convex metric space and C a closed convex subset of X. Then* $P_C(x) \subset V_C(x)$ *for each* $x \in X$.

Proof. The result is trivial if $P_C(x) = \phi$. If $x \in C$, then clearly $P_C(x) = \{x\}$ and $V_C(x) = \{x\}$ by Lemma 1. Now suppose $x \in X \diagdown C$ and let $x_o \in P_C(x)$. Then $x_o \in C$ and $x_o \neq x$. If $[x, x_o[\cap C \neq \phi$, then there exists a z_λ for $0 < \lambda < 1$ such that

$$d(x, z_\lambda) = (1 - \lambda) d(x, x_o) < d(x, x_o).$$

This is a contradiction to the fact that x_o is a best approximation to x in C. Therefore $[x, x_o[\cap C = \phi$ and hence $x_o \in V_C(x)$. □

4 Visible Points in Linear Metric Spaces

The following result shows that in linear metric spaces, the visible set mapping V_C satisfies translation property which is also satisfied by the set mapping P_C (see [7]).

Lemma 2. *Let (X, d) be a linear metric space, C a closed convex set and $x, y \in X$. Then $V_C(x) = V_{C+y}(x + y) - y$.*

Proof. For $x, y \in X$ and $v \in C$,

$$\begin{aligned} v \in V_C(x) &\Leftrightarrow [x, v[\cap C = \phi \\ &\Leftrightarrow [x + y, v + y[\cap (C + y) = \phi \\ &\Leftrightarrow v + y \in V_{C+y}(x + y) \\ &\Leftrightarrow v \in V_{C+y}(x + y) - y. \end{aligned}$$

□

Next, we explore those closed convex sets C having the property $V_C(x) = C$ for each $x \notin C$ i.e. we find answer to the question: For which sets is the whole set visible from any point outside the set? For this we recall the following notion(see [2]):

A set A of a linear metric space (X, d) is said to be **affine** if the line through each pair of points in A lies in A i.e. if the line

aff$\{\alpha_1 a_1 + \alpha_2 a_2 : \alpha_1 + \alpha_2 = 1\} \subset A$ for each pair a_1, $a_2 \in A$. Equivalently, A is affine if and only if $A = M + a$ for some unique linear subspace M and $a \in A$.

The following result gives the class of sets C for which $V_C(x) = C$.

Theorem 4. *Let C be a closed convex set in a linear metric space (X, d). Then the following statements are equivalent:*

1. *C is affine.*
2. *$V_C(x) = C$ for each $x \in X \diagdown C$.*

Proof. (1)$\Rightarrow$ (2). Let us first assume that $C = M$ is actually a subspace i.e. $0 \in C$. Fix any $x \notin M$. Since $V_M(x) \subset M$, it is sufficient to show that $M \subset V_M(x)$. For this, let

$m \in M$. If $m \notin V_M(x)$, then $[x, m[\cap M \neq \phi$. Hence there exists λ, $0 < \lambda < 1$ such that $\lambda x + (1-\lambda)m \in M$. Since $m \in M$, $\lambda x \in M$ and hence $x \in M$, a contradiction. This proves (2) in case C is a subspace . In general , suppose C is affine. Then $C = M + c$ for some subspace M and $c \in C$. For any $x \in X \setminus C$, we see that $x - c \notin M$ and by the above proof and Lemma 2 we obtain

$$V_C(x) = V_{M+c}(x) = V_M(x-c) + c = M + c = C.$$

(2) $\Rightarrow$ (1). Assume (2) holds . If C is not affine, then there exist distinct points $c_1, c_2 \in C$ such that aff$\{c_1, c_2\} \nsubseteq C$. Since C is closed convex and aff$\{c_1, c_2\}$ is a line, it follows that either aff$\{c_1, c_2\} \cap C = [y_1, y_2]$ for some distinct points y_1, y_2 in C. or aff$\{c_1, c_2\} \cap C = y_1 + \{\rho(y_2 - y_1) \mid \rho \geq 0\}$ for some distinct points y_1, y_2 in C. Therefore $x = y_1 + \rho(y_2 - y_1) \notin C$ for $-1 \leq \rho < 0$. But $y_1 = \frac{1}{1-\rho}x - \rho y_2 \in [x, y_2[\cap C$ proves that $y_2 \notin V_C(x)$. This contradicts the hypothesis as $V_C(x) = C$ if $x \notin C$. Hence C is affine. $\square$

Remarks: In the Euclidean space $\mathbb{R}^2$, if S is the unit sphere, then $P_S(0) = S$. It will be interesting to explore the class of sets C for which $P_C(x) = C$ for each $x \in X \setminus C$.

Acknowledgement: The authors are thankful to the learned referee for valuable comments and suggestions leading to an improvement of the paper.

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The Aligarh Bulletin of Mathematics
Volume 37, Numbers 1-2 (2018), 47-58
ISSN: 0304-9787

On common fixed point theorems for rational contractions in b-metric spaces

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(Received April 30, 2019)

Abstract

In this paper, we establish some fixed point and common fixed point theorems for rational contractions in the setting of b-metric spaces. Also, as a consequence, some results of integral type for such class of mappings is obtained. Our results extend and generalize several known results from the existing literature.

1 Introduction and Preliminaries

Fixed point theory plays a very significant role in the development of nonlinear analysis. In this area, the first important result was proved by Banach in 1922 for contraction mapping in complete metric space, known as the Banach contraction principle [3].

In [2], Bakhtin introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous contraction principle in metric spaces. Czerwik used the concept of b-metric space and

Keywords and phrases : Common fixed point, rational contraction, b-metric space
2010 AMS Subject Classification : 47H10, 54H25

generalized the renowned Banach fixed point theorem in b-metric spaces (see, [5, 6]).

In this note, we establish some fixed point and common fixed point theorems satisfying rational inequality in the framework of b-metric spaces.

Definition 1.1. ([1]) Let X be a nonempty set and let $d\colon X \times X \to R_+$ be a function satisfying the conditions:

$(A1)$ $d(x, y) = 0 \Leftrightarrow x = y$;

$(A2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;

$(A3)$ $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a metric on X and the pair (X, d) is called a metric space.

Definition 1.2. ([2]) Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d\colon X \times X \to R_+$ is called a b-metric if for all x, y, $z \in X$, the following conditions are satisfied:

$(B1)$ $d(x, y) = 0 \Leftrightarrow x = y$;

$(B2)$ $d(x, y) = d(y, x)$;

$(B3)$ $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space.

It is clear from the definition of b-metric space that every metric space is a b-metric space for $s = 1$. Therefore, the class of b-metric spaces is larger than the class of metric spaces.

Example 1.3. *([1]) Let $X = \{0, 1, 2\}$. Define $d\colon X \times X \to \mathbb{R}_+$ as follows $d(0, 0) = d(1, 1) = d(2, 2) = 0$, $d(1, 2) = d(2, 1) = d(0, 1) = d(1, 0) = 1$, $d(2, 0) = d(0, 2) = p \geq 2$ for $s = \frac{p}{2}$ where $p \geq 2$, the function defined as above is a b-metric space but not a metric space for $p > 2$.*

Example 1.4. *([7]) Let $X = \ell^p$ with $0 < p < 1$, where $\ell^p = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Let $d\colon X \times X \to \mathbb{R}$ defined by $d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$, where $x = \{x_n\}$, $y = \{y_n\} \in \ell^p$. Then (X, d) is a b-metric space with the coefficient $s = 2^{\frac{1}{p}} > 1$, since by an elementary calculation, we get that $d(x, y) \leq 2^{\frac{1}{p}}[d(x, z) + d(z, y)]$, but it is not a metric space.*

Example 1.5. *([7]) Let $X = \{1, 2, 3, 4\}$. Define $d\colon X \times X \to \mathbb{R}^2$ by*

$$d(x, y) = \begin{cases} (|x-y|^{-1}, |x-y|^{-1}) & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then (X, d) is a b-metric space with the coefficient $s = \frac{6}{5} > 1$. But it is not a metric space since the triangle inequality is not satisfied,

$$d(1,2) > d(1,4) + d(4,2), \quad d(3,4) > d(3,1) + d(1,4).$$

In our main result we will use the following definitions which can be found in [1] and [8].

Definition 1.6. Let (X, d) be a b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

$(C1)$ $\{x_n\}$ is a Cauchy sequence whenever, if for $\varepsilon > 0$, there exists a positive integer N such that for all $n, m \geq N$, $d(x_n, x_m) < \varepsilon$;

$(C2)$ $\{x_n\}$ is called convergent if for $\varepsilon > 0$ and $n \geq N$, we have $d(x_n, x) < \varepsilon$, where x is called the limit point of the sequence $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

$(C3)$ (X, d) is said to be a complete b-metric space if every Cauchy sequence in X converges to a point in X.

Remark 1.7. In a b-metric space (X, d), the following assertions hold:

(i) a convergent sequence has a unique limit;

(ii) each convergent sequence is Cauchy;

(iii) in general, a b-metric is not continuous.

2 Main Results

In this section we shall prove some fixed point and common fixed point theorems for rational contraction in the framework of b-metric spaces.

Theorem 2.1. *Let (X, d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mappings S, $T\colon X \to X$ satisfy:*

$$d(Sx, Ty) \quad \leq \quad k\left[\frac{d(x,Sx)d(x,Ty) + [d(x,y)]^2 + d(x,Sx)d(x,y)}{d(x,Sx) + d(x,y) + d(x,Ty)}\right] \tag{2.1}$$

for all $x, y \in X$, $k \in [0, 1)$ with $sk < 1$ and $d(x, Sx) + d(x, y) + d(x, Ty) \neq 0$. Then S and T have a common fixed point in X. Further if $d(x, Sx) + d(x, y) + d(x, Ty) = 0$ implies that $d(Sx, Ty) = 0$. Then S and T have a unique common fixed point in X.

Proof. Choose $x_0 \in X$. Let $x_1 = S(x_0)$ and $x_2 = T(x_1)$ such that

$$x_{2n+1} = Sx_{2n}, \; x_{2n+2} = Tx_{2n+1}, \; n = 0, 1, 2, \ldots.$$

Let $d(x, Sx) + d(x, y) + d(x, Ty) \neq 0$. Then from (2.1), we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\leq k\Big[\Big(d(x_{2n}, Sx_{2n})d(x_{2n}, Tx_{2n+1}) + [d(x_{2n}, x_{2n+1})]^2 \\
&\quad + d(x_{2n}, Sx_{2n})d(x_{2n}, x_{2n+1})\Big) \\
&\quad \times \Big(d(x_{2n}, Sx_{2n}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, Tx_{2n+1})\Big)^{-1}\Big] \\
&= k\Big[\Big(d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2}) + [d(x_{2n}, x_{2n+1})]^2 \\
&\quad + d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1})\Big) \\
&\quad \times \Big(d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2})\Big)^{-1}\Big] \\
&= k\, d(x_{2n}, x_{2n+1}) \\
&\quad \times \Big[\frac{d(x_{2n}, x_{2n+2}) + 2d(x_{2n}, x_{2n+1})}{d(x_{2n}, x_{2n+2}) + 2d(x_{2n}, x_{2n+1})}\Big] \\
&= k\, d(x_{2n}, x_{2n+1}). \qquad (2.2)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
d(x_{2n}, x_{2n+1}) &= d(Sx_{2n-1}, Tx_{2n}) \\
&\leq k\Big[\Big(d(x_{2n-1}, Sx_{2n-1})d(x_{2n-1}, Tx_{2n}) + [d(x_{2n-1}, x_{2n})]^2 \\
&\quad + d(x_{2n-1}, Sx_{2n-1})d(x_{2n-1}, x_{2n})\Big) \\
&\quad \times \Big(d(x_{2n-1}, Sx_{2n-1}) + d(x_{2n-1}, x_{2n}) + d(x_{2n-1}, Tx_{2n})\Big)^{-1}\Big] \\
&= k\Big[\Big(d(x_{2n-1}, x_{2n})d(x_{2n-1}, x_{2n+1}) + [d(x_{2n-1}, x_{2n})]^2 \\
&\quad + d(x_{2n-1}, x_{2n})d(x_{2n-1}, x_{2n})\Big) \\
&\quad \times \Big(d(x_{2n-1}, x_{2n}) + d(x_{2n-1}, x_{2n}) + d(x_{2n-1}, x_{2n+1})\Big)^{-1}\Big] \\
&= k\, d(x_{2n-1}, x_{2n}) \\
&\quad \times \Big[\frac{d(x_{2n-1}, x_{2n+1}) + 2d(x_{2n-1}, x_{2n})}{d(x_{2n-1}, x_{2n+1}) + 2d(x_{2n-1}, x_{2n})}\Big] \\
&= k\, d(x_{2n-1}, x_{2n}). \qquad (2.3)
\end{aligned}$$

By induction, we have

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq k\, d(x_n, x_{n-1}) \leq k^2\, d(x_{n-1}, x_{n-2}) \leq \dots \\
&\leq k^n\, d(x_1, x_0). \qquad (2.4)
\end{aligned}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned}
d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\
&= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\
&\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\
&= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\
&\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\
&\quad + \cdots + s^{n+m-1} d(x_{n+m-1}, x_m) \\
&\leq [sk^n + s^2 k^{n+1} + s^3 k^{n+2} + \cdots + s^m k^{n+m-1}] d(x_1, x_0) \\
&= sk^n [1 + sk + s^2k^2 + s^3k^3 + \cdots + (sk)^{m-1}] d(x_1, x_0) \\
&\leq \Big[\frac{sk^n}{1 - sk}\Big] d(x_1, x_0).
\end{aligned}$$

Since $0 < sk < 1$, therefore taking limit $m, n \to \infty$, we have

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in complete b-metric space X. Since X is complete, so there exists $p \in X$ such that $\lim_{n\to\infty} x_n = p$. Now, we have to show that p is a common fixed point of S and T. For this consider

$$\begin{aligned}
d(x_{2n+1}, Tp) &= d(Sx_{2n}, Tp) \\
&\leq k\Big[\frac{d(x_{2n}, Sx_{2n})d(x_{2n}, Tp) + [d(x_{2n}, p)]^2 + d(x_{2n}, Sx_{2n})d(x_{2n}, p)}{d(x_{2n}, Sx_{2n}) + d(x_{2n}, p) + d(x_{2n}, Tp)}\Big] \\
&= k\Big[\frac{d(x_{2n}, x_{2n+1})d(x_{2n}, Tp) + [d(x_{2n}, p)]^2 + d(x_{2n}, x_{2n+1})d(x_{2n}, p)}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, p) + d(x_{2n}, Tp)}\Big].
\end{aligned}$$

Taking limit $n \to \infty$, we have

$$d(p, Tp) \leq 0.$$

Thus $Tp = p$, that is, p is a fixed point of T.

In an exactly the same fashion we can prove that $Sp = p$. Hence $Sp = Tp = p$. This shows that p is a common fixed point of S and T.

Uniqueness

Let q be another common fixed point of S and T, that is, $Sq = Tq = q$ such that $p \neq q$. Suppose that $d(x, Sx) + d(x, y) + d(x, Ty) = 0$ implies that $d(Sx, Ty) = 0$. Now, we take $x = p$ and $y = q$ in the hypothesis, we have

$$d(p, Sp) + d(p, q) + d(p, Tq) = 0 \Rightarrow d(Sp, Tq) = 0.$$

Therefore, we get

$$d(p,q) = d(Sp,Tq) = 0.$$

Hence $p = q$. This shows that p is a unique common fixed point of S and T. This completes the proof. $\square$

Putting $S = T$ in Theorem 2.1, then we have the following result.

Corollary 2.2. *Let (X,d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$d(Tx,Ty) \leq k\Big[\frac{d(x,Tx)d(x,Ty) + [d(x,y)]^2 + d(x,Tx)d(x,y)}{d(x,Tx) + d(x,y) + d(x,Ty)}\Big] \tag{2.5}$$

for all $x,y \in X$, $k \in [0,1)$ with $sk < 1$ and $d(x,Tx) + d(x,y) + d(x,Ty) \neq 0$. Then T has a fixed point in X. Further if $d(x,Tx) + d(x,y) + d(x,Ty) = 0$ implies that $d(Tx,Ty) = 0$. Then T has a unique fixed point in X.

Proof. The proof of corollary 2.2 is immediately follows from Theorem 2.1 by taking $S = T$. This completes the proof. $\square$

Corollary 2.3. *Let (X,d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies (for fixed n):*

$$d(T^nx,T^ny) \leq k\Big[\frac{d(x,T^nx)d(x,T^ny) + [d(x,y)]^2 + d(x,T^nx)d(x,y)}{d(x,T^nx) + d(x,y) + d(x,T^ny)}\Big] \tag{2.6}$$

for all $x,y \in X$, $k \in [0,1)$ with $sk < 1$ and $d(x,T^nx) + d(x,y) + d(x,T^ny) \neq 0$. Then T has a fixed point in X. Further if $d(x,T^nx) + d(x,y) + d(x,T^ny) = 0$ implies that $d(T^nx,T^ny) = 0$. Then T has a unique fixed point in X.

Proof. By Corollary 2.2, there exists $v \in X$ such that $T^nv = v$. Then

$$\begin{aligned}
d(Tv,v) &= d(TT^nv,T^nv) = d(T^nTv,T^nv) \\
&\leq k\Big[\frac{d(Tv,T^nTv)d(Tv,T^nv) + [d(Tv,v)]^2 + d(Tv,T^nTv)d(Tv,v)}{d(Tv,T^nTv) + d(Tv,v) + d(Tv,T^nv)}\Big] \\
&= k\Big[\frac{d(Tv,TT^nv)d(Tv,T^nv) + [d(Tv,v)]^2 + d(Tv,TT^nv)d(Tv,v)}{d(Tv,TT^nv) + d(Tv,v) + d(Tv,T^nv)}\Big] \\
&= k\Big[\frac{d(Tv,Tv)d(Tv,v) + [d(Tv,v)]^2 + d(Tv,Tv)d(Tv,v)}{d(Tv,Tv) + d(Tv,v) + d(Tv,v)}\Big] \\
&= \frac{k}{2}\,d(Tv,v) \\
&\leq k\,d(Tv,v).
\end{aligned}$$

The above inequality is possible only if $d(Tv, v) = 0$ and so $Tv = v$. This shows that T has a unique fixed point in X. This completes the proof. $\square$

Theorem 2.4. *Let (X, d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T \colon X \to X$ satisfies the contraction condition:*

$$d(Tx, Ty) \leq \beta \max\Big\{d(x, y), \frac{d(x, Tx), d(y, Ty)}{1 + d(x, y)}, \frac{d(x, Tx), d(y, Ty)}{1 + d(Tx, Ty)}\Big\} \tag{2.7}$$

for all $x, y \in X$, $\beta \in [0, 1)$ is a constant with $s\beta < 1$. Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (2.7), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \beta \max\Big\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)}{1 + d(x_{n-1}, x_n)}, \\ &\qquad \frac{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)}{1 + d(Tx_{n-1}, Tx_n)}\Big\} \end{aligned} \tag{2.8}$$

$$\begin{aligned} &= \beta \max\Big\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n), d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)}, \\ &\qquad \frac{d(x_{n-1}, x_n), d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})}\Big\} \\ &\leq \beta \max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\}. \end{aligned} \tag{2.9}$$

If $\max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\} = d(x_n, x_{n+1})$, then from (2.9), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta\, d(x_n, x_{n+1}) \\ &< \frac{1}{s} d(x_n, x_{n+1}) \\ &< d(x_n, x_{n+1}), \end{aligned} \tag{2.10}$$

which is a contradiction.

Hence $\max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\} = d(x_{n-1}, x_n)$, so from (2.9), we have

$$d(x_n, x_{n+1}) \leq \beta\, d(x_{n-1}, x_n). \tag{2.11}$$

By induction, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta\, d(x_{n-1}, x_n) \leq \beta^2\, d(x_{n-2}, x_{n-1}) \leq \dots \\ &\leq \beta^n\, d(x_0, x_1). \end{aligned} \tag{2.12}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\ &\quad + \dots + s^{n+m-1} d(x_{n+m-1}, x_m) \\ &\leq [s\beta^n + s^2\beta^{n+1} + s^3\beta^{n+2} + \dots + s^m\beta^{n+m-1}] d(x_1, x_0) \\ &= s\beta^n[1 + s\beta + s^2\beta^2 + s^3\beta^3 + \dots + (s\beta)^{m-1}] d(x_1, x_0) \\ &\leq \Big[\frac{s\beta^n}{1 - s\beta}\Big] d(x_1, x_0). \end{aligned}$$

Since $0 < s\beta < 1$, therefore taking limit $m, n \to \infty$, we have

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in complete b-metric space X. Since X is complete, so there exists $q \in X$ such that $\lim_{n\to\infty} x_n = q$. Now, we have to show that q is a fixed point of T. For this consider

$$\begin{aligned} d(x_{n+1}, Tq) &= d(Tx_n, Tq) \\ &\leq \beta \max\Big\{d(x_n, q), \frac{d(x_n, Tx_n), d(q, Tq)}{1 + d(x_n, q)}, \frac{d(x_n, Tx_n), d(q, Tq)}{1 + d(Tx_n, Tq)}\Big\} \\ &= \beta \max\Big\{d(x_n, q), \frac{d(x_n, x_{n+1}), d(q, Tq)}{1 + d(x_n, q)}, \frac{d(x_n, x_{n+1}), d(q, Tq)}{1 + d(x_{n+1}, Tq)}\Big\}. \end{aligned}$$

Taking the limit $n \to \infty$, we have

$$d(q, Tq) \leq 0.$$

Thus $Tq = q$, that is, q is a fixed point of T.

Uniqueness

Let q' be another fixed point of T, that is, $Tq' = q'$ such that $q \neq q'$, then from (2.7), we have

$$\begin{aligned} d(q,q') &= d(Tq,Tq') \\ &\leq \beta \max\Big\{d(q,q'), \frac{d(q,Tq),d(q',Tq')}{1+d(q,q')}, \frac{d(q,Tq),d(q',Tq')}{1+d(Tq,Tq')}\Big\} \\ &= \beta \max\Big\{d(q,q'), \frac{d(q,q),d(q',q')}{1+d(q,q')}, \frac{d(q,q),d(q',q')}{1+d(q,q')}\Big\} \\ &= \beta \max\Big\{d(q,q'),0,0\Big\} \\ &\leq \beta\, d(q,q'). \end{aligned}$$

The above inequality is possible only if $d(q,q') = 0$ and so $q = q'$. Thus q is a fixed point of T. This completes the proof. $\square$

If $\max\Big\{d(x,y), \frac{d(x,Tx),d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx),d(y,Ty)}{1+d(Tx,Ty)}\Big\} = d(x,y)$, then from Theorem 2.4, we have the following result as corollary.

Corollary 2.5. *Let (X,d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$d(Tx,Ty) \leq \beta\, d(x,y)$$

for all $x,y \in X$, where $\beta \in (0,1)$ is a constant with $s\beta < 1$. Then T has a unique fixed point in X.

Remark 2.6. Corollary 2.5 extends well known Banach contraction principle from complete metric space to that setting of complete b-metric space considered in this paper.

We give an example in support of Theorems 2.1 and 2.4 as follows.

Example 2.7. *Let $X = [0,\infty)$ be endowed with b-metric and $d(x,y) = |x-y|^2 = (x-y)^2$, where $s = 2$. We consider the the mappings S, $T\colon X \to X$ defined by $S(x) = ln(1+\frac{x}{3})$ and $T(x) = ln(1+\frac{x}{2})$. Observe that $S(X) = T(X) = [0,\infty)$. For each $x,y \in X$ with $x \neq y$, we have*

$$\begin{aligned} d(Sx,Ty) &= (Sx-Ty)^2 = [ln(1+\frac{x}{3}) - ln(1+\frac{y}{2})]^2 \\ &< \Big(\frac{x}{3}-\frac{y}{2}\Big)^2, \ \textit{since } ln(1+x) < x \\ &= \frac{1}{36}(2x-3y)^2 \\ &\leq \frac{1}{36}(4x-4y)^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{16}{36}(x-y)^2 \\
&= \frac{4}{9}\, d(x,y) \\
&\leq \frac{4}{9}\Big[\frac{d(x,Sx)d(x,Ty)+[d(x,y)]^2+d(x,Sx)d(x,y)}{d(x,Sx)+d(x,y)+d(x,Ty)}\Big]
\end{aligned}
$$

where $\frac{4}{9} \leq k < 1$ and $s = 2$. Thus S and T satisfy all the conditions of Theorem 2.1. Moreover 0 is the unique common fixed point of S and T.

Example 2.8. *Let $X = [0,\infty)$ be endowed with b-metric and $d(x,y) = |x-y|^2 = (x-y)^2$, where $s = 2$. We consider the the mapping $T\colon X \to X$ defined by $T(x) = \frac{x}{2}$. Observe that $T(X) = [0,\infty)$. For each $x, y \in X$ with $x \neq y$, we have*

$$
\begin{aligned}
d(Tx,Ty) &= (Tx-Ty)^2 = \Big(\frac{x}{2}-\frac{y}{2}\Big)^2 \\
&= \frac{1}{4}(x-y)^2 \\
&\leq \frac{1}{2}\,(x-y)^2 \\
&= \frac{1}{2}\, d(x,y) \\
&\leq \frac{1}{2}\,\max\Big\{d(x,y), \frac{d(x,Tx),d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx),d(y,Ty)}{1+d(Tx,Ty)}\Big\}
\end{aligned}
$$

where $\beta = \frac{1}{2} < 1$ and $s = 2$. Thus T satisfies all the conditions of Theorem 2.4. Moreover 0 is the unique fixed point of T.

Other consequences of our results for the mappings involving contractions of integral type are the following.

Denote Λ the set of functions $\varphi\colon [0,\infty) \to [0,\infty)$ satisfying the following hypothesis:

(h1) φ is a Lebesgue-integrable mapping on each compact subset of $[0,\infty)$;

(h2) for any $\varepsilon > 0$ we have $\int_0^{\varepsilon} \varphi(t)dt > 0$.

Theorem 2.9. *Let (X,d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mappings S, $T\colon X \to X$ satisfy the condition:*

$$
\int_0^{d(Sx,Ty)} \psi(t)dt \quad \leq \quad k \int_0^{\Big[\frac{d(x,Sx)d(x,Ty)+[d(x,y)]^2+d(x,Sx)d(x,y)}{d(x,Sx)+d(x,y)+d(x,Ty)}\Big]} \psi(t)dt
$$

for all $x, y \in X$, where $k \in [0,1)$ is a constant with $sk < 1$ and $\psi \in \Lambda$. Then S and T have a unique common fixed point in X.

If we put $S = T$ in Theorem 2.9, then we have the following result.

Theorem 2.10. *Let (X, d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mappings $T\colon X \to X$ satisfies the condition:*

$$\int_0^{d(Tx,Ty)} \psi(t)dt \quad \leq \quad k \int_0^{\left[\frac{d(x,Tx)d(x,Ty)+[d(x,y)]^2+d(x,Tx)d(x,y)}{d(x,Tx)+d(x,y)+d(x,Ty)}\right]} \psi(t)dt$$

for all $x, y \in X$, where $k \in [0,1)$ is a constant with $sk < 1$ and $\psi \in \Lambda$. Then T has a unique fixed point in X.

Theorem 2.11. *Let (X, d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$\int_0^{d(Tx,Ty)} \psi(t)dt \quad \leq \quad \beta \int_0^{\max\left\{d(x,y), \frac{d(x,Tx),d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx),d(y,Ty)}{1+d(Tx,Ty)}\right\}} \psi(t)dt$$

for all $x, y \in X$, where $\beta \in [0,1)$ is a constant with $s\beta < 1$ and $\psi \in \Lambda$. Then T has a unique fixed point in X.

Theorem 2.12. *Let (X, d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$\int_0^{d(Tx,Ty)} \psi(t)dt \quad \leq \quad \beta \int_0^{d(x,y)} \psi(t)dt$$

for all $x, y \in X$, where $\beta \in [0,1)$ is a constant with $s\beta < 1$ and $\psi \in \Lambda$. Then T has a unique fixed point in X.

Remark 2.13. Theorem 2.12 extends Theorem 2.1 of Branciari [4] from complete metric space to that setting of complete b-metric space for integral type contraction considered in this paper.

3 Conclusion

In this paper, we establish some unique fixed point and common fixed point theorems for rational contractions in the setting of b-metric spaces. Also, as a consequence, we obtain some results of integral type contraction for such mappings. Our results extend and generalize several results from the existing literature.

4 Acknowledgement

The author would like to thanks the anonymous referee for his careful reading and useful suggestions on the manuscript.

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The Aligarh Bulletin of Mathematics
Volume 37, Numbers 1-2 (2018), 59-70

ISSN: 0304-9787

On $*$- Ideals and derivations in prime rings with involution

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(Received June 25, 2019)

Abstract

Let $\mathscr{R}$ be a ring with involution $'*'$. An additive map $x \mapsto x^*$ of $\mathscr{R}$ into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ holds for all $x, y \in \mathscr{R}$. An additive mapping $\delta : \mathscr{R} \to \mathscr{R}$ is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathscr{R}$. The purpose of this paper is to examine the commutativity of prime rings with involution satisfying certain identities involving derivations.

1 Introduction and Notations

In all that follows, unless specially stated, $\mathscr{R}$ always denotes an associative ring with centre $\mathcal{Z}(\mathscr{R})$. As usual the symbols $s \circ t$ and $[s, t]$ will denote the anti-commutator $st + ts$ and commutator $st - ts$, respectively. Given an integer $n \geq 2$, a ring $\mathscr{R}$ is said to be n-torsion free if $nx = 0$ (where $x \in \mathscr{R}$) implies that $x = 0$. A ring $\mathscr{R}$ is called prime if $a\mathscr{R}b = (0)$ (where $a, b \in \mathscr{R}$) implies $a = 0$ or $b = 0$ and is called semiprime ring if $a\mathscr{R}a = (0)$ (where $a \in \mathscr{R}$) implies $a = 0$. An additive map $x \mapsto x^*$ of $\mathscr{R}$ into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii)

Keywords and phrases : Prime ring, $*$-ideal, involution, derivation
2010 AMS Subject Classification : 16N60; 16W10; 16W25

$(x^*)^* = x$ hold for all $x, y \in \mathscr{R}$. A ring equipped with an involution is called ring with involution or $*$-ring. An ideal I of R is said to be $*$-ideal of R if $I^* = I$. An element x in a ring with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of $\mathscr{R}$ will be denoted by $H(\mathscr{R})$ and $S(\mathscr{R})$, respectively. The involution is called the first kind if $\mathcal{Z}(\mathscr{R}) \subseteq H(\mathscr{R})$, otherwise it is said to be of the second kind. In the later case $S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R}) \neq (0)$. Notice that in case x is normal i.e., $xx^* = x^*x$, if and only if h and k commute. If all elements in $\mathscr{R}$ are normal, then $\mathscr{R}$ is called a normal ring(see [15] for more details).

An additive mapping $\delta : \mathscr{R} \to \mathscr{R}$ is said to be a derivation of $\mathscr{R}$ if $\delta(st) = \delta(s)t + s\delta(t)$ for all $s, t \in \mathscr{R}$. A derivation δ is said to be inner if there exists $a \in \mathscr{R}$ such that $\delta(s) = as - sa$ for all $s \in \mathscr{R}$. For an automorphism α, an additive mapping $\delta : \mathscr{R} \to \mathscr{R}$ is said to be a skew derivation of $\mathscr{R}$ if $\delta(st) = \delta(s)t + \alpha(s)\delta(t)$ for all $s, t \in \mathscr{R}$. Over the last some decades, several authors have investigated the relationship between the commutativity of the ring $\mathscr{R}$ and certain special types of additive maps like derivations, skew derivations and automorphisms of $\mathscr{R}$. The criteria to discuss the commutativity of prime rings via derivations has been studied first time by Posner [23]. In fact, he proved that the existence of a nonzero centralizing derivation($i.e.,$ $\delta(x)x - x\delta(x) \in \mathcal{Z}(\mathscr{R})$ for all $x \in \mathcal{Z}(\mathscr{R})$ on a prime ring forces that the ring to be commutative. Since then many algebraists studied the commutativity of prime and semiprime rings via derivations, skew derivations or automorphisms that satisfying certain identities (viz.; [2, 4, 5, 6, 7, 8, 13, 14, 19, 21] and references therein). In [9], Bell and Daif showed that if R is a prime ring admitting a nonzero derivation δ such that $\delta(st) = \delta(ts)$ for all $s, t \in \mathscr{R}$, then $\mathscr{R}$ is commutative. This result was extended for semiprime rings by Daif [11]. In 2016, S. Ali et. al [3], studied these results in the setting of rings with involution involving derivations(see also [12]).

In this paper, our intent is to continue this line of investigation and discuss the commutativity of prime rings with involution involving derivations in more general situation .

2 The results

We shall do a great deal of calculation with commutators and anti-commutators, routinely using the following basic identities:

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = [x, y]z + y[x, z] \text{ for all } x, y, z \in R.$$

Moreover

$$xo(yz) = (xoy)z - y[x,z] = y(xoz) + [x,y]z \text{ for all } x,y,z \in R.$$

and

$$(xy)oz = (xoz)y + x[y,z] = x(yoz) - [x,z]y \text{ for all } x,y,z \in R.$$

We start our investigation with some well known facts and results in rings which will be used frequently throughout the text.

Fact 2.1. *If $\mathscr{R}$ is a prime ring and $0 \neq b \in \mathcal{Z}(\mathscr{R})$ and $ab \in \mathcal{Z}(\mathscr{R})$, then $a \in \mathcal{Z}(\mathscr{R})$.*

Fact 2.2. *If a prime ring $\mathscr{R}$ contains a nonzero central ideal, then $\mathscr{R}$ is commutative.*

Fact 2.3. *Let $\mathscr{R}$ be a prime ring and I be a nonzero $*$-ideal of $\mathscr{R}$. If x is an element of I such that $[x,[y,z^*]] = 0$ for all $y,z \in I$, then $x \in \mathcal{Z}(\mathscr{R})$.*

Proof. Substituting yw for y in the given condition, we obtain

$$[x,y][w,z^*] + [y,z^*][x,w] = 0 \text{ for all } y,z,w \in I.$$

In particular, for $w = x$, we have $[x,y][x,z^*] = 0$. Replacing y by ry, where $r \in \mathscr{R}$, we get $[x,r]y[x,z^*] = 0$ for all $y,z \in I$ and $r \in \mathscr{R}$. This implies that $[x,\mathscr{R}] = \{0\}$ or $I[x,I] = \{0\}$. In both the cases, we can conclude that $x \in \mathcal{Z}(\mathscr{R})$. □

Fact 2.4. *Let $\mathscr{R}$ be a 2-torsion free ring with involution $'*'$. Then every $x \in \mathscr{R}$ can be uniquely represented as $2x = h + k$, where $h \in H(\mathscr{R})$ and $k \in S(\mathscr{R})$.*

Fact 2.5. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of second kind such that $char(\mathscr{R}) \neq 2$. Let δ be a nonzero derivation of $\mathscr{R}$ such that $\delta(h) = 0$ for all $h \in S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Then $\delta(x) = 0$ for all $x \in \mathcal{Z}$.*

Proof. By the assumption, we have $\delta(h) = 0$ for all $h \in S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Substituting k^2(where $k \in S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$) for h, we get $\delta(k^2 = \delta(k)k + k\delta(k)$. This implies that $2k\delta(k) = 0$ for all $k \in S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Application of Fact 2.1 yields $\delta(k) = 0$ for all $k \in S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. In view of Fact 2.4, we conclude that $2\delta(x) = \delta(2x) = \delta(h+k) = \delta(h) + \delta(k) = 0$ and hence $\delta(x) = 0$ for all $x \in \mathcal{Z}$. □

We begin with the following lemmas:

Lemma 2.1. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of R. If $[x,x^*] \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Proof. Linearization of the relation $[x, x^*] \in \mathcal{Z}(\mathscr{R})$ gives that

$$[x, y^*] + [y, x^*] \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.1}$$

Replacing y by yk, where $k \in S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$, we get

$$-[x, y^*]k + [y, x^*]k \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I \text{ and } k \in S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R}). \tag{2.2}$$

Combining (2.1) and (2.2), we obtain $2[y, x^*]k \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$ and $k \in S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Since $\mathscr{R}$ is a prime ring of $char(\mathscr{R}) \neq 2$ with second kind involution, the above relation gives $[y, x^*] \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. This implies that $[y, x] \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. This implies that I is commutative and hence $\mathscr{R}$ is commutative. □

Lemma 2.2. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of R. If $x \circ x^* \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Proof. Linearize the given condition, we have

$$x \circ y^* + y \circ x^* \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.3}$$

Replacing x by xk, where $k \in S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$, we get

$$(x \circ y^*)k - (y \circ x^*)k \in Z(\mathscr{R}) \text{ for all } x, y \in I \text{ and } k \in S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R}). \tag{2.4}$$

Multiplying by k to (2.3) and combining the obtained expression with (2.4), we obtain $2(x \circ y^*)k \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$ and $k \in S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Since $\mathscr{R}$ is a prime ring of $char(\mathscr{R}) \neq 2$, we obtain $x \circ y^* \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. This implies that $x \circ y \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. That is

$$[x \circ y, r] = 0 \text{ for all } x, y \in I \text{ and } r \in \mathscr{R}. \tag{2.5}$$

Replacing x by xt in (2.5), we get

$$(y \circ x)[t, r] + y[[t, x], r] + [y, r][t, x] = 0 \text{ for all } x, y \in I \text{ and } r, t \in \mathscr{R}.$$

In particular, for $t = x$ the above relation reduces as $(y \circ x)[x, r] = 0$ for all $x, y \in I$ and $r \in \mathscr{R}$. In view of Fact 2.1, either $y \circ x = 0$ or $[x, r] = 0$ for all $x, y \in I$ and $r \in \mathscr{R}$. In both the cases, we conclude that $\mathscr{R}$ is commutative. □

Lemma 2.3. *Let $\mathscr{R}$ be a prime ring of $char(\mathscr{R}) \neq 2$ and I be nonzero ideal of $\mathscr{R}$. If $\mathscr{R}$ admits a nonzero derivation δ such that $[\delta(x), \delta(y)] \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Proof. By the assumption, we have

$$[\delta(x), \delta(y)] \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I.$$

Since I and $\mathscr{R}$ satisfy the same differential identities (see [17, Theorem 2]), then we have

$$[\delta(x), \delta(y)] \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{R}.$$

Thus, in view of [18, Theorem 2] $\mathscr{R}$ must be commutative. □

Corollary 2.1. *Let $\mathscr{R}$ be a prime ring of $char(\mathscr{R}) \neq 2$ and I be a nonzero ideal of $\mathscr{R}$. If δ is a nonzero derivation of $\mathscr{R}$ such that $\delta(x)\delta(y) \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Theorem 2.1. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a nonzero derivation of $\mathscr{R}$ such that $[\delta(x), \delta(x^*)] \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Proof. By the assumption, we have

$$[\delta(x), \delta(x^*)] \in \mathcal{Z}(\mathscr{R}) \text{ for all } x \in I. \tag{2.6}$$

On linearization of (2.6), we get

$$[\delta(x), \delta(y^*)] + [\delta(y), \delta(x^*)] \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.7}$$

Replacing y by yh (where $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$) in (2.7), we obtain

$$([\delta(x), y^*] + [y, \delta(x^*)])\delta(h) \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I.$$

Fact 2.1 gives that $[\delta(x), y^*] + [y, \delta(x^*)] \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$ or $\delta(h) = 0$ for all $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$. First, we consider the case

$$[\delta(x), y^*] + [y, \delta(x^*)] \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.8}$$

Substitute y by yk (where $k \in \mathcal{Z}(\mathscr{R}) \cap S(\mathscr{R})$) in above relation, we obtain

$$(-[\delta(x), y^*] + [y, \delta(x^*)])k \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I.$$

Using Fact 2.1 and the condition $Z(\mathscr{R}) \cap S(\mathscr{R}) \neq \{0\}$, we get

$$-[\delta(x), y^*] + [y, \delta(x^*)] \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.9}$$

Adding (2.8) and (2.9), we get $2[y, \delta(x^*)] \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. This implies that $[x, \delta(x)] \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$. Hence $\mathscr{R}$ is commutative (see [10, Theorem

4]). Now, consider the second case $\delta(h) = 0$ for all $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$, and hence $\delta(z) = 0$ for all $z \in \mathcal{Z}(\mathscr{R})$ by the Fact 2.5. Replacing y by yk (where $k \in \mathcal{Z}(\mathscr{R}) \cap S(\mathscr{R})$) in (2.7), we get

$$(-[\delta(x), \delta(y^*)] + [\delta(y), \delta(x^*)])k \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I.$$

This implies that

$$-[\delta(x), \delta(y^*)] + [\delta(y), \delta(x^*)] \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.10}$$

Combining (2.7) and (2.10), we obtain $[\delta(y), \delta(x^*)] \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. The last expression gives $[\delta(y), \delta(x)] \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. Thus in view of Lemma 2.3, $\mathscr{R}$ is a commutative Integral domain. □

Corollary 2.2. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a nonzero derivation of $\mathscr{R}$ such that $\delta(x)\delta(x^*) \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$ or $\delta(x^*)\delta(x) \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Corollary 2.3. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a nonzero derivation of $\mathscr{R}$ such that $[\delta(x), \delta(x^*)] + x \circ x^* \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Proof. By the assuption, we have $[\delta(x), \delta(x^*)] + x \circ x^* \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$. Interchanging the role of x and x^* and using the fact that $[x, x^*] = -[x^*, x]$, we find that $[\delta(x), \delta(x^*)] - x \circ x^* \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$. On combining the last two relations, we get $[\delta(x), \delta(x^*)] \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$. Application of Theorem 2.1 gives the required result.

□

In [1], first author together with N. A. Dar proved the following theorem.

Theorem 2.2. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ such that $char(\mathscr{R}) \neq 2$. Let δ be a nonzero derivation of $\mathscr{R}$ such that $\delta([x, x^*]) = 0$ for all $x \in \mathscr{R}$ and $S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R}) \neq (0)$. Then $\mathscr{R}$ is commutative.*

In the following theorem, we prove the same result in a more general setting.

Theorem 2.3. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a nonzero derivation of $\mathscr{R}$ such that $\delta([x, x^*]) \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Proof. By the hypothesis, we have

$$\delta([x,x^*]) \in \mathcal{Z}(\mathscr{R}) \text{ for all } x \in I. \tag{2.11}$$

Substituting $x+y$ for x in (2.11), we get

$$\delta([x,y^*]) + \delta([y,x^*]) \in \mathcal{Z}(\mathscr{R}) \text{ for all } x,y \in I. \tag{2.12}$$

Replacing y by yh (where $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$) in (2.12) and using it, we obtain

$$([x,y^*] + [x,y^*])\delta(h) \in \mathcal{Z}(\mathscr{R}) \text{ for all } x,y \in I. \tag{2.13}$$

Taking $x = y$ in (2.13), we arrive at

$$2[x,x^*]\delta(h) \in \mathcal{Z}(\mathscr{R}) \text{ for all } x \in I. \tag{2.14}$$

Since $char(\mathscr{R}) \neq 2$, so the last relation gives $[x,x^*]\delta(h) \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$. In view of Fact 2.5, we have either $[x,x^*] \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$ or $\delta(h) = 0$ for all $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$. If $[x,x^*] \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then by Lemma 2.1, $\mathscr{R}$ is commutative.
On the other hand if $\delta(h) = 0$ for all $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$, then $\delta(k) = 0$ for all $k \in \mathcal{Z}(\mathscr{R}) \cap S(\mathscr{R})$. Replacing y by yk in (2.11) and using it, we get

$$2\delta([y,x^*])k \in \mathcal{Z}(\mathscr{R}) \text{ for all } x,y \in I.$$

This implies that $\delta([y,x^*]) \in \mathcal{Z}(\mathscr{R})$ for all $x,y \in I$. By taking $x = x^*$, we have $\delta([y,x]) \in \mathcal{Z}(\mathscr{R})$ for all $x,y \in I$. By [4, Theorem 3.12], $\mathscr{R}$ is commutative. This completes the proof of the theorem. □

Corollary 2.4. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a nonzero derivation of $\mathscr{R}$ such that $\delta(xx^*) \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$ or $\delta(x^*x) \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Corollary 2.5. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a nonzero derivation of $\mathscr{R}$ such that $\delta([x,x^*]) + x \circ x^* \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Theorem 2.4. *Let $\mathscr{R}$ be a prime ring with involution$'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a nonzero derivation of $\mathscr{R}$ such that $\delta(x \circ x^*) \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Proof. By the assumption, we have

$$\delta(x \circ x^*) \in \mathcal{Z}(\mathscr{R}) \text{ for all } x \in I.$$

Linearize the above relation, we have

$$\delta(x \circ y^*) + \delta(y \circ x^*) \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.15}$$

Substitute $y = yh$ in (2.15), where $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$, we obtain

$$(x \circ y^* + y \circ x^*)\delta(h) \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I.$$

This implies that $x \circ y^* + y \circ x^* \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$ or $\delta(h) = 0$ for all $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$. Firstly, we consider the case $x \circ y^* + y \circ x^* \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. In particular for $x = y$, we have $2(x \circ x^*) \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$. Application of Lemma 2.2 yields the required result.
On the other hand if $\delta(h) = 0$ for all $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$, then by Fact 2.5 we conclude that $\delta(k) = 0$ for all $k \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$. Replacing y by yk in (2.15), we get

$$(-\delta(x \circ y^*) + \delta(y \circ x^*))k \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I.$$

This implies that

$$-\delta(x \circ y^*) + \delta(y \circ x^*) \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.16}$$

Adding (2.15) and (2.16), we obtain $2\delta(y \circ x^*) \in \mathcal{Z}(\mathscr{R})$. Hence $\delta(y \circ x) \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. Therefore $\mathscr{R}$ is commutative in view of [22, Theorem 7]. □

Corollary 2.6. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a nonzero derivation of $\mathscr{R}$ such that $\delta(x \circ x^*) + [x, x^*] \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Theorem 2.5. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a derivation of $\mathscr{R}$ such that $\delta([x, x^*]) + [x, x^*] \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Proof. By the assumption, we have

$$\delta([x, x^*]) + [x, x^*] \in \mathcal{Z}(\mathscr{R}) \text{ for all } x \in I. \tag{2.17}$$

If $\delta = 0$, then by Lemma 2.1 $\mathscr{R}$ is commutative. Now, assume that $\delta \neq 0$. Linearization of (2.17) gives that $\delta([x, y^*]) + \delta([y, x^*]) + [x, y^*] + [y, x^*] \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. Taking $y = y^*$, we get

$$\delta([x, y]) + \delta([y^*, x^*]) + [x, y] + [y^*, x^*] \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.18}$$

Replacing y by yh in (2.18) and using it, we obtain $([x, y] + [y^*, x^*])\delta(h) \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. By Fact 2.1, we have either $[x, y] + [y^*, x^*] \in \mathcal{Z}(\mathscr{R})$ or $\delta(h) = 0$. Consider the first case

$$[x, y] + [y^*, x^*] \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.19}$$

Taking yk for y in (2.19) (where $k \in \mathcal{Z}(\mathscr{R}) \cap S(\mathscr{R})$, we get

$$([x, y] - [y^*, x^*])k \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.20}$$

Multiplying by k to (2.19) and combining it with the obtained relation, we get $[x, y]k \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. This implies that $[x, y] \in Z(R)$ for all $x, y \in I$. By the Fact 2.3, $I \subseteq \mathcal{Z}(\mathscr{R})$. Therefore in view of Fact 2.2, $\mathscr{R}$ is commutative.
Now, consider the second case $\delta(h) = 0$ for all $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$. By Fact 2.5, we have $\delta(z) = 0$ for all $z \in \mathcal{Z}(\mathscr{R})$. Replacing y by yk in (2.18) using the fact that $\delta(k) = 0$, we obtain

$$\delta([x, y]) - \delta([y^*, x^*]) + [x, y] - [y^*, x^*] \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.21}$$

Adding (2.18) and (2.21), we get $\delta([x, y]) + [x, y] \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. Thus in view of [16, Theorem 1] $\mathscr{R}$ is commutative. $\square$

Similarly, we can prove the following:

Theorem 2.6. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a derivation of $\mathscr{R}$ such that $\delta([x, x^*]) - [x, x^*] \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Corollary 2.7. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a derivation of $\mathscr{R}$ such that $\delta([x, x^*]) + x \circ x^* \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Theorem 2.7. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a derivation of $\mathscr{R}$ such that $\delta(x \circ x^*) + x \circ x^* \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Proof. By the assumption, we have

$$\delta(x \circ x^*) + x \circ x^* \in \mathcal{Z}(\mathscr{R}) \text{ for all } x \in I.$$

If $\delta = 0$, then by Lemma 2.2 $\mathscr{R}$ is commutative. Now, assume that $\delta \neq 0$. Linearization of the above condition gives that

$$\delta(x \circ y^*) + \delta(y \circ x^*) + x \circ y^* + y \circ x^* \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.22}$$

Putting yh for y in (2.22), where $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$, we get

$$(x \circ y^* + y \circ x^*)\delta(h) \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I.$$

This implies that $(x \circ y^* + y \circ x^*) \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$ or $\delta(h) = 0$. Consider the first case $(x \circ y^* + y \circ x^*) \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. In particular for $x = y$, we have $2(x \circ x^*) \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$. Since $char(\mathscr{R}) \neq 2$ and in view of Lemma 2.2, $\mathscr{R}$ is commutative.
Now, consider the second case $\delta(h) = 0$ for all $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$. By the Fact 2.5, we have $\delta(z) = 0$ for all $z \in \mathcal{Z}(\mathscr{R})$. Replacing y by yk in (2.22), where $k \in \mathcal{Z}(\mathscr{R}) \cap S(\mathscr{R})$, we obtain

$$(-\delta(x \circ y^*) + \delta(y \circ x^*) - (x \circ y^*) + y \circ x^*)k \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I.$$

Since $\mathcal{Z}(\mathscr{R}) \cap S(\mathscr{R}) \neq \{0\}$, and in view of Fact 2.1, we conclude that

$$-\delta(x \circ y^*) + \delta(y \circ x^*) - (x \circ y^*) + y \circ x^* \in \mathcal{Z}(\mathscr{R}) \text{ for all } x, y \in I. \tag{2.23}$$

Adding (2.22) and (2.23), we obtain $\delta(y \circ x^*) + y \circ x^* \in \mathcal{Z}(\mathscr{R})$. This implies that $\delta(y \circ x) + y \circ x \in \mathcal{Z}(\mathscr{R})$ for all $x, y \in I$. Therefore $\mathscr{R}$ is commutative in view of [22, Theorem 10]. This completes the proof of the theorem. □

Similarly we can prove the following theorem:

Theorem 2.8. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a derivation of $\mathscr{R}$ such that $\delta(x \circ x^*) - x \circ x^* \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Corollary 2.8. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a derivation of $\mathscr{R}$ such that $\delta(xx^*) + xx^* \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

Corollary 2.9. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and I be a nonzero $*$-ideal of $\mathscr{R}$. If δ is a derivation of $\mathscr{R}$ such that $\delta(xx^*) - xx^* \in \mathcal{Z}(\mathscr{R})$ for all $x \in I$, then $\mathscr{R}$ is a commutative Integral domain.*

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In the list of reference, the following examples should be observed:

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Vol. 37 Numbers 1-2 (2018)