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HYDROMAGNETIC FLOW OF A TWO-PHASE FLUID THROUGH POROUS MEDIUM NEAR A PULSATING PLATE

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Abstract

An initial value investigation is made of the motion of an incompressible, viscous, conducting fluid through porous medium with embedded small inert spherical particles bounded by an infinite rigid non-conducting plate. The unsteady flow is supposed to generate from rest in the fluid-particle system due to velocity tooth pulses being imposed on the plate in presence of a transverse magnetic field. It is assumed that no external electric field is acting on the system and the magnetic Reynolds number is very small. The operational method is used to obtain exact solutions for the fluid and the particle velocities and the shear stress at the plate. Quantitative analysis of the results is made to disclose the simultaneous effects of the magnetic field, porosity of porous medium and the particles on the fluid velocity and the wall shear stress.

1 Introduction

The fluid flow generated by the pulsatile motion of the boundary is found have immense importance in aerospace science, nuclear fusion, astrophysics, atmospheric science, cosmical gasdynamics, seismology and physiological fluid dynamics. The investigation in this direction was initiated by Ghosh [5] who examined the motion of an incompressible viscous fluid in a channel bounded by two coaxial circular cylinders when the inner cylinder is set in motion by pulses of longitudinal impulses. Subsequently, Chakraborty and Ray [2] studied the unsteady magneto hydrodynamic couette flow between two parallel plates when one of the plates is subjected to random pulses. Makar [10] presented the solution of magnetohydrodynamic flow between two parallel plates when the velocity tooth pulses are imposed on the upper plate and the induced magnetic field is neglected. Bestman and Njoku

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AMS Subject Classification : 76R10, 76W05.

[1] constructed the solution of the same problem as that of author [10] without ignoring the effect of the induced magnetic field and using the methodology different from that of author [10] to arrive at the solution of the problem. Most recently, Ghosh and Debnath [6] considered the hydro magnetic channel flow of a dusty fluid induced by tooth pulses while Ghosh and Ghosh [7] solved the same problem as that of authors [6] replacing the boundary condition at the upper plate of the channel by rectified sine pulses instead of tooth pulses as encountered by authors [6]. On the other hand, Datta et al. [3, 4] examined the heat transfer to pulsatile flow of a dusty fluid in pipes and channel with a view to their applications in the analysis of blood flow. Recently, Ghosh and Ghosh [8] have studied on hydromagnetic flow of a two-phase fluid near a pulsating plate. In spite of the above works it is noticed that the development of the unsteady flow in a semi-infinite expanse of fluid due to pulsatile motion of the boundary has hardly received any attention although such problems are important for the analysis of suspension boundary layers. The main objective of this paper is to study these problems with a view to physical applications.

The present paper is concerned with the unsteady hydromagnetic flow of a semi-infinite expanse of an incompressible, electrically conducting, and viscous fluid through porous medium containing uniformly distributed small inert spherical particles bounded by an infinite rigid non-conducting plate. The motion is supposed to generate from rest in the fluid-particle system due to velocity tooth pulses imparted on the plate. The analysis is carried out to obtain exact solutions for the fluid through porous medium and the particle velocities and the shear stress exerted by the fluid on the pulsating plate. The quantitative analysis is made to examine the effects of the particles and the magnetic field, porosity of porous medium on the fluid velocity and the wall shear stress.

2 Mathematical formulation

Based upon the two-phase fluid flow model of Saffman [11], the equations of unsteady motion of an electrically conducting viscous fluid through porous medium with embedded identical small inert spherical particles in presence of an external magnetic field are in usual notation.

$$\frac{\partial u}{\partial t} + (u.\Delta)u = -\frac{1}{\rho}\Delta p + v\Delta^2 u + \frac{kN}{\rho}(v-u) + \frac{1}{\rho}(j \times B) - \frac{v}{K}u \tag{2.1}$$

$$m\left[\frac{\partial v}{\partial t} + (v.\Delta)v\right] = k(u-v) \tag{2.2}$$

$$\Delta u = 0 \quad \text{and} \quad \frac{\partial N}{\partial t} + \Delta.(Nv) = 0 \tag{2.3}$$

Where

$u = (u_x, u_y, u_z) =$ fluid velocity

$v = (v_x, v_y, v_z) =$ particle velocity

$B = (B_x, B_y, B_z)$ = magnetic flux density

$j = (j_x, j_y, j_z)$ = Current density

p =fluid pressure

N = number density of the particles

ρ, v = density and kinematic viscosity of the fluid

m = mass of the individual particles

k = Stokes resistance coefficient which for spherical particles (of radius a is $6\pi\mu a$)

K = permeability of porous medium

In the above set of equations the particles are assumed sufficiently small so that gravitational action on them in equation (2.2) may be neglected compared with the fluid velocity.

The Maxwell equations with usual MHD approximations are:

$$div\ B = 0, \quad rot\ B = \mu j, \quad rot\ E = -\frac{\partial B}{\partial t} \tag{2.4}$$

Where

$E = (E_x, E_y, E_z)$ =electric field

$j = \sigma(E + u \times B)$

σ = electrical conductivity

μ = magnetic permeability

We take x-axis in the direction of flow with origin at the plate and y-axis perpendicular to the plate. The motion is generated in the fluid-particle system due to velocity tooth pulses imposed on the plate. Is the strength of the external magnetic field acting parallel to y-axis. Since the motion is a plane one and the plate is infinitely long, we assume that all the physical variables are independent of x and z. then from the equations (2.3) of continuity and from the physical condition of the problem, we have

$$u = [u_x(y,t), 0, 0], \quad v = [v_x(y,t), 0, 0], \quad N = N_0 = \text{ constant} \tag{2.6}$$

Further from the first equation of (4), $\frac{\partial B_y}{\partial y} = 0$ gives

$$B_y = \text{ Constant } = B_0 \tag{2.7}$$

It is also obvious from the physical situation that and will vanish. Second equation of then gives

$$j_y = 0 \qquad \text{and} \qquad \mu j_z = -\frac{\partial B_x}{\partial y} \tag{2.8}$$

Again the fluid flows in the x-direction and there is no external electric field, E can have z-component only.

It therefore follows from (2.5) that

$$(j \times B)_x = -\sigma B_0(E_z + u_x B_0) \tag{2.9}$$

We assume at this stage that σ is small so that the perturbation in the magnetic field may be neglected. We also assume that the current is mainly due to the induced electric field $j = \sigma(u \times B)$ so that E_z can be neglected. Therefore, from equations (2.1), (2.2) and (2.9), we have

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial y^2} + \partial k\tau(v-u) - \frac{\sigma B_0^2}{\rho}u - \frac{v}{K}u \tag{2.10}$$

and

$$\frac{\partial u}{\partial t} = \frac{1}{\tau}(u-v) \tag{2.11}$$

Where (u_x, v_x) are replaced by (u, v), $k = \frac{mN_0}{\rho}$ = ratio of the mass density of the particles and the fluid density = mass concentration of the particles and $\tau = \frac{m}{k}$ = Relaxation time of the particles.

Introducing the non-dimensional variables

$$u' = \frac{u}{U}, \qquad v' = \frac{v}{U}, \qquad y' = \frac{y}{\int v\tau}, \qquad t' = \frac{t}{\tau}, \qquad K' = \frac{vK}{\tau}$$

in (2.10) and (2.11) and dropping the primes, we get the non-dimensional equations in the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + k(v-u) - nu \tag{2.13}$$

and

$$\frac{\partial u}{\partial t} = (u-v) \tag{2.14}$$

for $0 \leq y \leq \infty, \quad t \geq 0$, where $n = (M + \frac{1}{K})$, $M = \frac{\sigma B_0^2 \tau}{\rho}$ represents the Hartman number.

The problem now reduces to solving equations (2.13) and (2.14) subject to the boundary conditions given by

$$u(y,t) = f(t) \quad \text{on } y = 0, \quad t > 0 \tag{2.15}$$

$$\{u(u,t), v(y,t)\} \to \{0,0\} \quad \text{as } y \to \infty, \qquad t > 0 \tag{2.16}$$

and the initial conditions

$$u(y,0) = 0 = v(y,0) \quad \text{for } 0 \le y \le \infty \tag{2.17}$$

Where $f(t)$ represents the tooth pulses which is an even periodic function with period 2 and strength ET.

3 Solution of the problem

In view of nature of $f(t)$ mentioned above the mathematical form of $u(0,t)$ may be written as

$$u(0,t) = \frac{E}{T}\left\{tH(t) + 2\sum^{\infty} p = 1(-1)^p(t - pT)H(t - pT)\right\} \tag{3.1}$$

Where $H(t)$ is the Heaviside unit step function defined as $H(t-T) = 0, t < T$ and $H(t-T) = 1, t \ge T$. By using half-range Fourier series the condition (3.1) may also be expressed as

$$u(0,y) = \frac{E}{2} - \frac{4E}{\pi^2}\sum^{\infty} p = 0\frac{1}{(2p+1)^2}\cos\left\{\frac{(2p+1)\pi t}{T}\right\} \tag{3.2}$$

The use of Laplace transforms method for the solution of (2.13) and (2.14) with initial condition (2.17) gives the transformed equation for the fluid velocity in the form:

$$\frac{d^2\bar{u}}{dy^2} - \left\{\frac{(1+s)(s+k+n) - k}{1+s}\right\}\bar{u} = 0 \tag{3.3}$$

With

$$\bar{u} \to 0 \quad \text{as } y \to \infty \tag{3.4}$$

And

$$\bar{u} = \frac{E}{Ts^2}\tan h\left(\frac{sT}{2}\right) mbox \quad at \quad y = 0 \tag{3.5}$$

Where s is the Laplace transform variable.
The transformed solution for the fluid velocity $\bar{u}(y,s)$ becomes,

$$\bar{u}(y,s) = \frac{E}{Ts^2}\tan h\left(\frac{sT}{2}\right) exp\left\{-y\left[\frac{(s+c)(s+d)}{s+1}\right]^{\frac{1}{2}}\right\} \tag{3.6}$$

Where

$$c = \frac{1}{2}\left[(a_1 + n) + \{a_1^2 + 2n(a_1 - 2) + n^2\}^{\frac{1}{2}}\right],$$

$$d = \frac{1}{2}\left[(a_1 + n) - \{a_1^2 + 2n(a_1 - 2) + n^2\}^{\frac{1}{2}}\right],$$

With

$$a_1 = 1 + k \quad \text{and} \quad c \geq a_1 \geq 1 > d.$$

The inversion of (3.6) gives

$$u(y,t) = \frac{E}{2\pi i T} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st} \sin h\left(\frac{sT}{2}\right)}{s^2 \cos h\left(\frac{sT}{2}\right)} exp\left\{-y\left[\frac{(s+c)(s+d)}{s+1}\right]^{\frac{1}{2}}\right\} ds \tag{3.7}$$

The inversion integral has a pole at $s = 0$ and a series of poles at $s = \pm i\beta_p$, $\beta_p = \frac{(2p+1)\pi}{T}$, $p = 0, 1, 2, \cdots$ and branch points at $s = -c, -d, -1$ as shown in the contour drawn in figure 1 in the complex s-plane.

Evaluating (3.7) with the help of Cauchy's residue theorem applied to the contour in figure 1, we get

$$\begin{aligned} \frac{u(y,t)}{E} &= \frac{1}{2} e^{-y\sqrt{cd}} - \frac{4}{T^2} \sum_{p=0}^{\infty} \frac{1}{\beta_p^2} exp\left\{\frac{yM_1}{\sqrt{2}}\right\} \cos\left[\frac{yM_2}{\sqrt{2}} - \beta_p t\right] \\ &- \frac{1}{\pi T} \int\limits_{c}^{\infty} \frac{\tan h \frac{xT}{2}}{x^2} e^{-xt} \sin\left\{y\sqrt{\frac{(x-c)(x-d)}{x-1}}\right\} dx \\ &- \frac{1}{\pi T} \int\limits_{d}^{1} \frac{\tan h \frac{xT}{2}}{x^2} e^{-xt} \sin\left\{y\sqrt{\frac{(c-x)(x-d)}{1-x}}\right\} dx \end{aligned} \tag{3.8}$$

Where $M_1, M_2 = \frac{1}{\sqrt{1+\beta_p^2}} \{\pm[cd + \beta_p^2(c + d - 1)]$

$$+ \sqrt{[cd + \beta_p^2(c + d - 1)]^2 + \beta_p^2[c + d - cd + \beta_p^2]^2}\}^{\frac{1}{2}}$$

It is to be noted here that when $E = 2$ and $T \to 0$ the result (3.8) coincides with the dimensionless form of the result corresponding to $\omega \to 0$ case of authors Yang and Healy [12] and describes the fluid velocity for hydro magnetic flow of a particulate suspension near an impulsively moved plate.

The particle velocity for the corresponding motion can be obtained from (2.14) as

$$v(y,t) = e^{-t} \int\limits_0^t u(y,\eta) e^{\eta} d\eta \tag{3.9}$$

which on using (3.8) becomes

$$\frac{v(y,t)}{E} = \frac{1}{2}e^{-y\sqrt{cd}}(1-e^{-1}) - \frac{4}{T^2}\sum_{p=0}^{\infty}\frac{1}{\beta_p^2\sqrt{1+\beta_p^2}}exp\left\{-\frac{yM_1}{\sqrt{2}}\right\}$$

$$\times\left\{\cos\left(\frac{yM_2}{\sqrt{2}}-\beta_p t\right) - e^{-1}\cos\left(\frac{yM_2}{\sqrt{2}}+\theta\right)\right\}$$

$$+\frac{1}{\pi T}\int_c^{\infty}\frac{\tan h\frac{xT}{2}}{x^2}\left\{\frac{e^{-xt}-e^{-1}}{x-1}\right\}\sin\left\{y\sqrt{\frac{(x-c)(x-d)}{x-1}}\right\}dx$$

$$+\frac{1}{\pi T}\int_d^{1}\frac{\tan h\frac{xT}{2}}{x^2}\left\{\frac{e^{-xt}-e^{-1}}{x-1}\right\}\sin\left\{y\sqrt{\frac{(c-x)(x-d)}{1-x}}\right\}dx \qquad (3.10)$$

Where

$$\theta = \tan^{-1}\beta_p$$

In particular when $k \to 0$, the result (3.8) provides the solution for the clean fluid velocity in the form

$$\frac{u(y,t)}{E} = \frac{1}{2}e^{-y\sqrt{n}} - \frac{4}{T^2}\sum_{p=0}^{\infty}\frac{1}{\beta_p^2}exp\left\{\frac{y\alpha_1}{\sqrt{2}}\right\}\cos\left[\frac{y\alpha_2}{\sqrt{2}}-\beta_p t\right]$$

$$-\frac{1}{\pi T}\int_n^{\infty}\frac{\tan h\frac{xT}{2}}{x^2}e^{-xt}\sin\left\{y\sqrt{(x-n)}\right\}dx \qquad (3.11)$$

Where

$$\alpha_1,\alpha_2 = \left\{\pm n + \sqrt{n^2+\beta_p^2}\right\}^{\frac{1}{2}}$$

Further, when $E = 2$ and $T \to 0$, (3.11) reduce to

$$u(y,t) = e^{-y\sqrt{n}} - \frac{1}{\pi}\int_n^{\infty}\frac{e^{-xt}}{x}\sin\{\sqrt{x-n}\}dx$$

$$= \frac{1}{2}\{e^{y\sqrt{n}}erfc(\eta+\sqrt{nt}) + e^{-y\sqrt{n}}erfc(\eta-\sqrt{nt})\}, \qquad \eta = \frac{y}{2\sqrt{t}} \qquad (3.12)$$

which is the well-known solution of hydro magnetic Rayleigh problem (cf. authors [12]).

On the other hand, if $n \to 0$ we get from (3.12) the classical Rayleigh solution as

$$u(y,t) = erfc(\eta) \qquad (3.13)$$

The fluid velocity given by (3.8) attains the steady-state in the limit $t \to \infty$ and the ultimate flow becomes

$$\frac{u(y,t)}{E} = \frac{1}{2}e^{-y\sqrt{n}} - \frac{4}{T^2}\sum_{p=0}^{\infty}\frac{1}{\beta_p^2}exp\left\{-\frac{yM_1}{\sqrt{2}}\right\}\cos\left[\frac{yM_2}{\sqrt{2}} - \beta_p t\right] \tag{3.14}$$

and the particle velocity in this situation is

$$\frac{v(y,t)}{E} = \frac{1}{2}e^{-y\sqrt{cd}} - \frac{4}{T^2}\sum_{p=0}^{\infty}\frac{exp\left(-\frac{yM_1}{\sqrt{2}}\right)}{\beta_p^2\sqrt{1+\beta_p^2}}\left\{\cos\left(\frac{yM_2}{\sqrt{2}} - \beta_p t + \theta\right)\right\} \tag{3.15}$$

Comparing (3.14) with (3.15) we find that the particles in the steady-state move faster than the fluid with a phase lead due to the presence of β_p. But when $\beta_p \to \infty$, i.e. $T \to 0$, we have $u = v$. This shows that the particles attain the fluid velocity in the steady motion generated by impulsively moved plate in an inertial system. This result is known from Michael and Miller's analysis [9]. Moreover, the ultimate flow given by (3.14) consists of two distinct boundary layers. One is a Hartman layer of thickness of the order $\sqrt{\frac{vT}{n}}$ and the other is a Stokes-Hartman layer of thickness of the order $\sqrt{\frac{2vT}{M_1}}$. Since $M_1 > n$ the thickness of the Hartman layer is greater than that of the Stokes-Hartman layer which decreases with the increase of the particles and the magnetic field. However, in the limit $T \to 0(\beta_p \to \infty)$ there exists only the classical Hartman layer in the vicinity of the plate.

The exact solution of the shear stress at the plate $y = 0$, in dimensionless form, is given by

$$\frac{\tau_0}{E} = \frac{\sqrt{cd}}{2} - \frac{4}{T^2}\sum_{p=0}^{\infty}\frac{M_1\cos\beta_p - t - M_2\sin\beta_p t}{\sqrt{2}\beta_p^2}$$
$$+\frac{1}{\pi T}\int_c^{\infty}\frac{\tan h\frac{xT}{2}}{x^2}e^{-xt}\left\{\sqrt{\frac{(x-c)(x-d)}{1-x}}\right\}dx$$
$$+\frac{1}{\pi T}\int_d^{1}\frac{\tan h\frac{xT}{2}}{x^2}e^{-xt}\left\{\sqrt{\frac{(c-x)(x-d)}{1-x}}\right\}dx \tag{3.16}$$

Which when $k \to 0$ yields

$$\frac{\tau_0}{E} = \frac{\sqrt{n}}{2} - \frac{4}{T^2}\sum_{p=0}^{\infty}\frac{\alpha_1\cos\beta_p - t - \alpha_2\sin\beta_p t}{\sqrt{2}\beta_p^2}$$
$$+\frac{1}{\pi T}\int_n^{\infty}\frac{\tan h\frac{xT}{2}}{x^2}e^{-xt}\left\{\sqrt{(x-n)}\right\}dx \tag{3.17}$$

However, when $E = 2$ and $T \to 0$, we have from (3.17)

$$\tau_0 = \sqrt{n} + \frac{1}{\pi}\int_n^{\infty}\frac{e^{-xt}}{\sqrt{x}}\sqrt{(x-n)}dx \tag{3.18}$$

Which is the shear stress at the wall corresponding to hydro magnetic Rayleigh problem and when $n \to 0$, provides the familiar result

$$\tau_0 = \frac{1}{\pi}\int_n^\infty \frac{e^{-xt}}{\sqrt{x}}dx = \frac{1}{\sqrt{(\pi t)}} \tag{3.19}$$

4 Numerical results

The quantitative analysis of the results (3.8) and (3.16) when $T = 2$ are presented in the figures (2) and (3). From figure - (2), it is seen that the magnetic field (M) has a diminishing effect on the flow which reduces with the increase of particles (k) in the fluid and enhances with the increase of time (t). For instance, when $t = 2.5$ and M increase from 0.01 to 0.1 the fluid velocity decreases when $k = 0$ and the fluid velocity increases with increasing porosity (K) of porous medium. Similar results for $t = 25$ they are respectively.

Regarding the effect of particles on the flow, we notice that, for small values of time when the effect of pulsation is reasonably small, the particles produce diminishing effect on the fluid velocity near the plate which is a consequence of inertia of the particles. But for large values of time, when the effect of pulsation is significantly high, the particles increase the fluid motion near the plate which is further enhanced with the increase of the magnetic field. In this situation, the particles attain the non-equilibrium process of relaxation due to the effect of pulsation. This stage continues up to a certain distance from the plate. As a result, the particulate motion near the plate cannot settle down as quickly as the clean fluid for large values of time. For example, if the unsteady motion is generated in a two-phase fluid system due to impulsively moved plate, the particles diminish the fluid velocity for all values of time owing to the effect of inertia of the particles playing a vital role to resist the motion (cf. Ref. 10). In this context, we would like to mention that the increasing effect produced by the particles in the fluid motion near the plate due to pulsation at large values of time can be controlled by introducing solid body rotation on the whole system. This phenomenon will be discussed in a subsequent paper of the authors.

Finally, we observed that the magnitude of the shear stress exerted by the fluid on the plate increases with the particles, the magnetic field and porosity which is expected up to moderately large values of time as shown in figure-(3).

The shear stress decreases with increasing porosity parameter (K). This figure further shows the appearance of negative shear stresses on the wall before the end and at the beginning of consecutive pulses acting on the plate. This is due to the fact that, during decelerating motion of the plate, the positive shear stress acting on it, exerted by the fluid, goes on diminishing and a stage will come when the fluid in motion drags the plate towards its destination by applying shear stress in opposite direction (negative shear stress) on it. Such a condition prevails till the plate acquires sufficient momentum to overcome the effect of the negative shear stress during its next accelerated motion caused by the pulse. It may also happen that there appears no negative shear stress on the plate if the strength of the magnetic field be further increased which damps the fluid motion sufficiently so that the plate can come to rest without the assistance of a negative shear stress produced by the fluid on it. This is consequence of the effect of the magnetic field on the motion of the fluid.

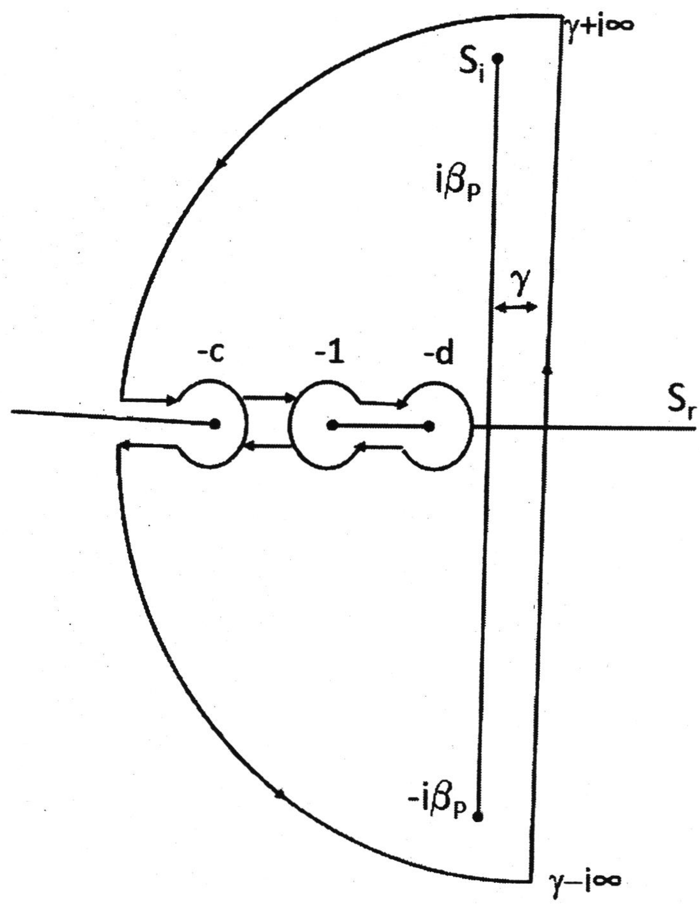

Fig. -1: Counter integral path for (3.7)

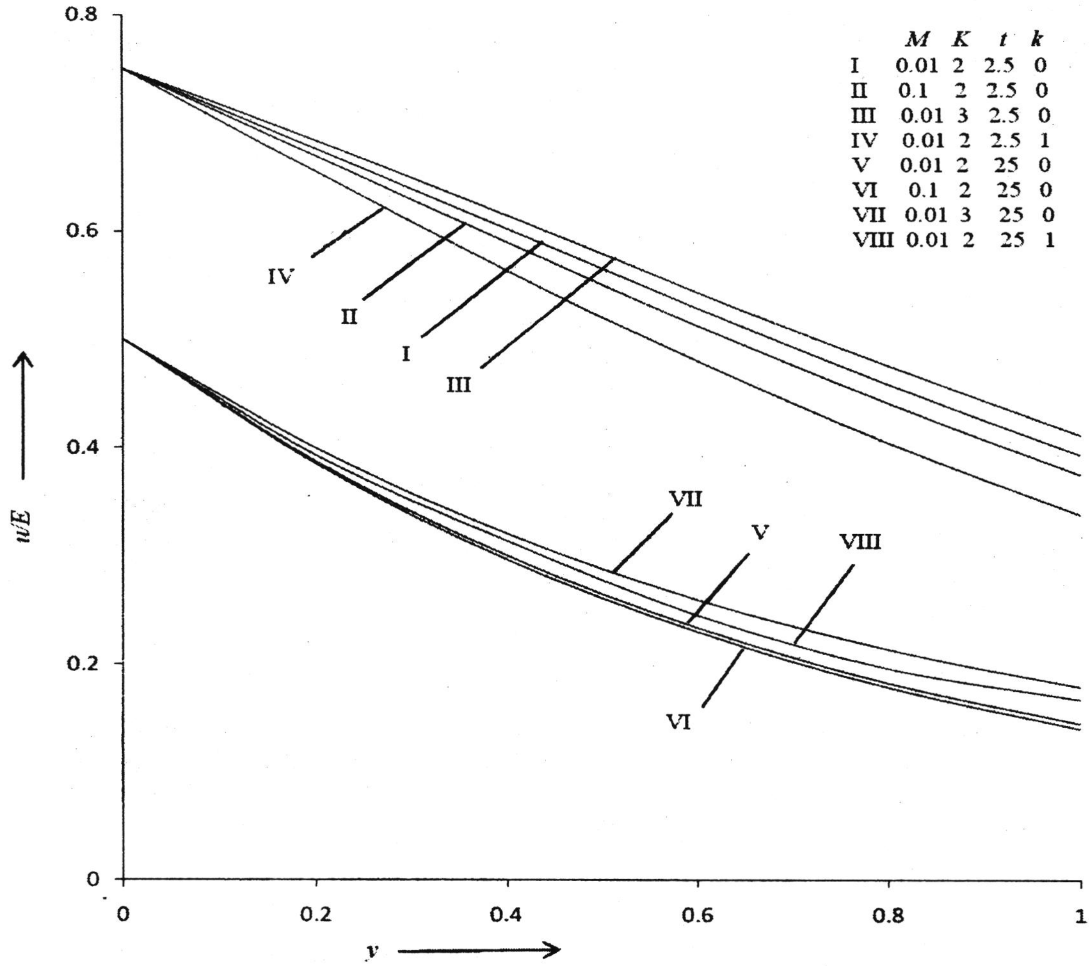

Fig.2: Velocity distribution for different values of M, K, t and k.

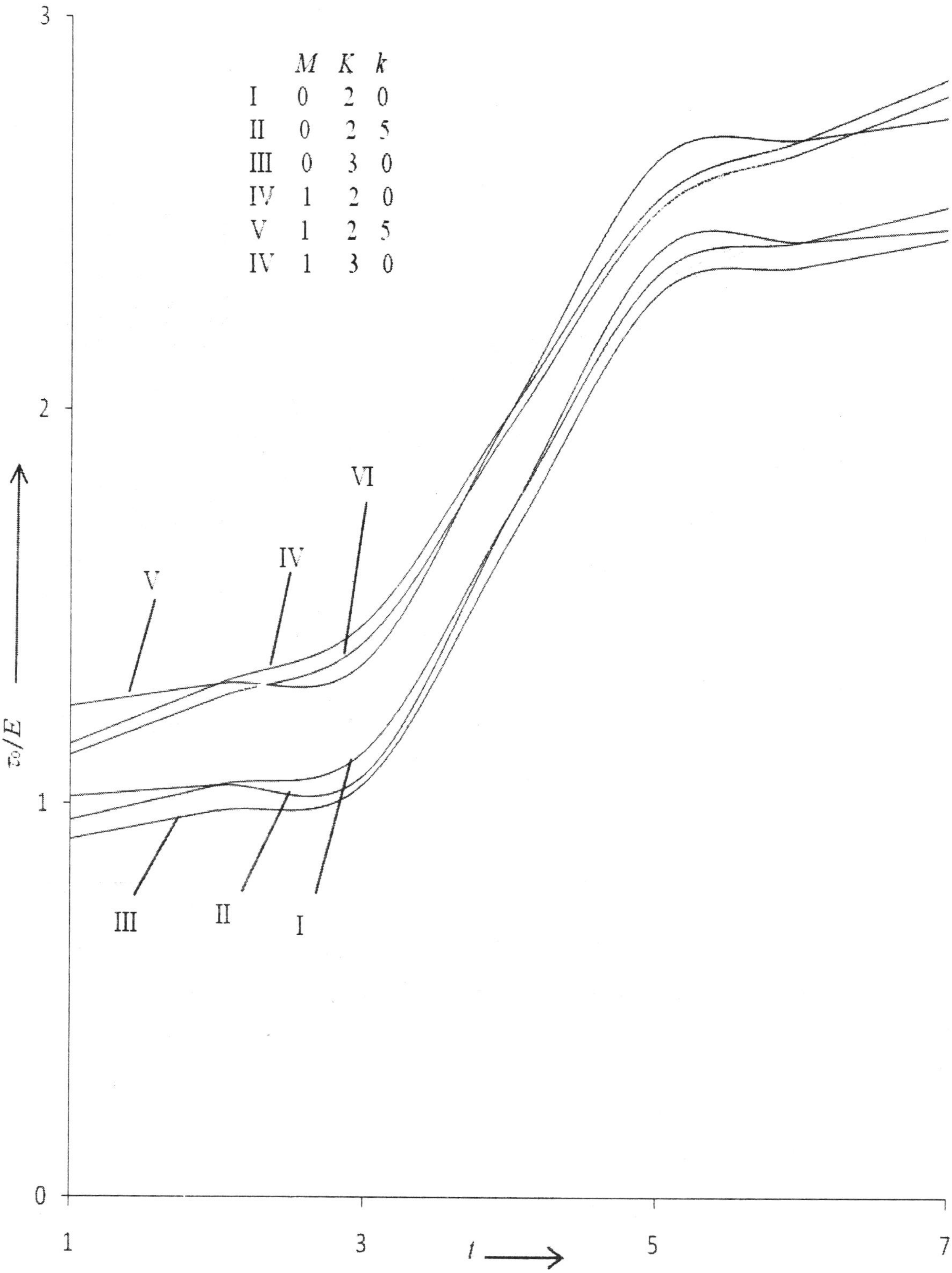

Fig.-3: Shear stress for different values of the particle concentration (k), M and K.

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A NEW METHOD FOR SOLVING FUZZY INTERVAL INTEGER TRANSPORTATION PROBLEM WITH MIXED CONSTRAINTS

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Abstract

In this paper we proposed a new method, namely Separation method based on zero method is used to find a solution of fuzzy interval integer transportation problem with mixed constraints. This method is very easy to understand and apply. The separation method can be served as an important tool for the decision makers when they are handling various types of logistic problems having interval parameters. This method can be illustrated with a numerical example.

1 Introduction

The transportation problem is to transport various amounts of a single homogeneous commodity that are initially stored at various origins to different destinations in such a way that the total transportation cost is minimum. It is a special class of a linear programming problem. Let us consider a production in which a transportation is from m-sources to n-destinations and their capacities $a_1, a_2, a_3, \cdots, a_m$ and $b_1, b_2, b_3, \cdots, b_n$ respectively. Various efficient methods were developed for solving transportation problems with the assumption of precise sources, destination parameters, and the penalty factors.

Many researchers see, e.g., [3, 4, 5] have solved transportation problems with inexact coefficients by fuzzy and interval programming techniques. Das and other researchers [5]

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proposed a method, called fuzzy technique to solve interval transportation problem by considering the right bound and the mid point of the interval. A new method have proposed by Sengupta, Pal [7] to solve Interval transportation problems with the mid point and width of the interval in the objective function.Adlakha et. al. [1] proposed a heuristic method for solving the transportation problems with mixed constraints which is based on the theory of shadow price. Recently, Pandian and Natarajan [6] have proposed a new algorithm for finding a fuzzy optimal solution for fuzzy transportation problem.

In this paper, we use separation method for finding optimal solution of fuzzy interval integer transportation problem with mixed constraints where all parameters are trapezoidal fuzzy numbers. This new method is based on zero method and also, it is very simple, easy to understand and apply. The solution procedure is illustrated with the help of numerical example.

2 Fuzzy integer transportation problem with mixed constraints

Consider the following fuzzy integer transportation problem with mixed constraints.

$$(P) \qquad min.\tilde{z} \approx \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij}\tilde{x}_{ij}$$

s.t.

$$\sum_{j=1}^{n} \tilde{x}_{ij} \approx \tilde{a}_i, i = 1, 2, 3, . \cdots, m$$

$$\sum_{j=1}^{n} \tilde{x}_{ij} \geq \tilde{a}_i, i = 1, 2, 3, . \cdots, m$$

$$\sum_{j=1}^{n} \tilde{x}_{ij} \leq \tilde{a}_i, i = 1, 2, 3, . \cdots, n$$

$$\sum_{i=1}^{m} \tilde{x}_{ij} \approx \tilde{b}_j, j = 1, 2, 3, . \cdots, n$$

$$\sum_{i=1}^{m} \tilde{x}_{ij} \geq \tilde{b}_j, j = 1, 2, 3, . \cdots, n$$

$$\sum_{i=1}^{m} \tilde{x}_{ij} \leq \tilde{b}_j, j = 1, 2, 3, . \cdots, n$$

$\tilde{x}_{ij} \geq \tilde{0}, i = 1, 2, 3, \cdots, m$ and $j = 1, 2, 3, \cdots, n$ and are integers
where

$$\tilde{z} = (z_1, z_2, z_3, z_4)$$

m = the number of supply points;

n = the number of demand points;

$\tilde{x}_{ij} = (x_y^1, x_y^2, x_y^3, x_y^4)$ is the uncertain number of units shipped from supply point i to demand point j;

$\tilde{c}_{ij} = (c_y^1, c_y^2, c_y^3, c_y^4)$ is the uncertain cost of shipping one unit from supply point i to the demand point j;

$\tilde{a}_i = (a_i^1, a_i^2, a_i^3, a_i^4)$ is the uncertain supply point i;

$\tilde{b}_i = (b_j^1, b_j^2, b_j^3, b_j^4)$ is the uncertain supply point j;

3 Fuzzy interval integer transportation problem with mixed constraints

A trapezoidal fuzzy number (a, b, c, d) can be converted as an interval number form as follows.

$$(a, b, c, d) = [a + (b - a)\alpha, d - (d - c)\alpha]; \alpha = 0 \text{ and } 1 \tag{3.1}$$

Using relation (3.1), we can convert the given fuzzy integer transportation problem with mixed constraints into a fuzzy interval integer transportation problem with mixed constraints. Such that

$$min.[z_1, z_2] = \sum_{i=1}^{m}\sum_{j=1}^{n}[c_{ij}^1, c_{ij}^2] \oplus [x_{ij}^1, x_{ij}^2]$$

s.t.

$$\sum_{j=1}^{n}[x_{ij}^1, x_{ij}^2] \approx [a_i^1, a_i^2], i = 1, 2, 3, \cdots, m;$$

$$\sum_{j=1}^{n}[x_{ij}^1, x_{ij}^2] \geq [a_i^1, a_i^2], i = 1, 2, 3, \cdots, m;$$

$$\sum_{j=1}^{n}[x_{ij}^1, x_{ij}^2] \leq [a_i^1, a_i^2], i = 1, 2, 3, \cdots, m;$$

$$\sum_{j=1}^{n}[x_{ij}^1, x_{ij}^2] \approx [b_j^1, b_j^2], j = 1, 2, 3, \cdots, n;$$

$$\sum_{j=1}^{n}[x_{ij}^1, x_{ij}^2] \geq [b_j^1, b_j^2], j = 1, 2, 3, \cdots, n;$$

$$\sum_{j=1}^{n}[x_{ij}^1, x_{ij}^2] \leq [b_j^1, b_j^2], j = 1, 2, 3, \cdots, n;$$

$x_{ij}^1 \geq 0, x_{ij}^2 \geq 0, i = 1, 2, 3, \cdots, m$ and $j = 1, 2, 3, \cdots, n$ and are integers.
where x_{ij}^1 and x_{ij}^2 are positive real numbers for all i and j, a_i^1 and a_i^2 are positive real numbers foı all i b_j^1 and b_j^2 are also positive real numbers for all j. Using the separation method, we can solve the interval transportation problem with zero method.

4 Upper and lower bound integer transportation problem of fuzzy interval integer transportation problem with mixed constraint

The upper bound integer transportation problem of the fuzzy interval integer transportation problem is

$$min.z_2 = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^2, x_{ij}^2$$

s.t.

$$\sum_{j=1}^{n} x_{ij}^2 \approx a_i^2, i = 1, 2, 3, \cdots, m;$$

$$\sum_{j=1}^{n} x_{ij}^2 \geq a_i^2, i = 1, 2, 3, \cdots, m;$$

$$\sum_{j=1}^{n} x_{ij}^2 \leq a_i^2, i = 1, 2, 3, \cdots, m;$$

$$\sum_{i=1}^{m} x_{ij}^2 \approx b_j^2, j = 1, 2, 3, \cdots, n;$$

$$\sum_{i=1}^{m} x_{ij}^2 \geq b_j^2, j = 1, 2, 3, \cdots, n;$$

$$\sum_{i=1}^{m} x_{ij}^2 \leq b_j^2, j = 1, 2, 3, \cdots, n;$$

Then the set $\{\bar{x}_i^2 j$ for all i and $j\}$ is an optimal solution of the upper bound integer transportation problem.

The lower bound integer transportation problem of the fuzzy interval integer transportation problem is

$$min.z_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^1, x_{ij}^1$$

s.t.

$$\sum_{j=1}^{n} x_{ij}^1 \approx a_i^1, i = 1, 2, 3, \cdots, m;$$

$$\sum_{j=1}^{n} x_{ij}^1 \geq a_i^1, i = 1, 2, 3, \cdots, m;$$

$$\sum_{j=1}^{n} x_{ij}^1 \leq a_i^1, i = 1, 2, 3, \cdots, m;$$

$$\sum_{i=1}^{m} x_{ij}^1 \approx b_j^1, j = 1, 2, 3, \cdots, n;$$

$$\sum_{i=1}^{m} x_{ij}^1 \geq b_j^1, j = 1, 2, 3, \cdots, n;$$

$$\sum_{i=1}^{m} x_{ij}^1 \leq b_j^1, j = 1, 2, 3, \cdots, n;$$

Then the set $\{\bar{x}_i^1 j$ for all i and $j\}$ is an optimal solution of the lower bound integer transportation problem.

5 Separation method

Separation method can be understood with the help of algorithm for solving fuzzy interval integer transportation problem. Algorithm of the separation method is as follows.

Step 1. Write the upper bound integer transportation problem of the given fuzzy interval integer transportation problem.

Step 2. Solve the upper bound integer transportation problem using zero method.

Step 3. Construct the lower bound integer transportation problem of the given fuzzy interval integer transportation problem.

Step 4. Solve the lower bound integer transportation problem using zero method.

Step 5. The solution of the given fuzzy interval integer transportation problem is $\{[\bar{x}_i^1 j, \bar{x}_i^2 j]$ for all i and $j\}$.

6 Zero method

We, now proposed a new method called zero method for finding the optimal solution for the transportation problem. The method is proceeding as follows.

Step 1. Convert all inequalities into equalities.

Step 2. If the given transportation is in unbalanced transportation problem, then make it balance transportation problem by introducing dummy rows or columns.

Step 3. Subtract each row entries of the transportation table from the corresponding row minimum.

Step 4. Subtract each column entries of the transportation table from the corresponding column minimum.

Step 5. Remember that each row and each column has at least one zero.

Step 6. Allocate the minimum cost from demand or supply in the corresponding zero.

Step 7. Repeat the procedure form the step 3 to step 6, until we get the optimal solution.

Step 8. Place the loads of the dummy rows or columns of the balanced at the lowest cost feasible cells of the given transportation problem to obtain the optimal solution for the transportation problem with mixed constraints.

Step 9. Thus we get the optimal solution for the new transportation problem with mixed constraints.

7 Numerical Example

Consider the following fuzzy integer transportation problem with mixed constraints.

Table 1

	1	2	3	Supply
1	(1,2,3,4)	(2,5,8,11)	(2,4,6,8)	$\approx$(2,5,8,11)
2	(2,6,10,14)	(1,3,5,7)	(0,1,2,3)	$\geq$(3,6,9,12)
3	(4,8,12,16)	(3,9,15,21)	(1,2,3,4)	$\leq (3,9,15,21)$
demand	$\approx$(4,8,12,16)	$\geq$(8,10,12,14)	$\leq$(3,5,7,9)	

Now, the fuzzy interval integer transportation problem of the above problem is given below.

Table 2

	1	2	3	Supply
1	$(1+\alpha, 4-\alpha)$	$(2+3\alpha, 11-3\alpha)$	$(2+2\alpha, 8-2\alpha)$	$\approx (2+3\alpha 5, 11-3\alpha)$
2	$(2+4\alpha, 14-4\alpha)$	$(1+2\alpha, 7-2\alpha)$	$(0+\alpha, 3-\alpha)$	$\geq (3+3\alpha, 12-3\alpha)$
3	$(4+4\alpha, 16-4\alpha)$	$(3+6\alpha, 21-6\alpha)$	$(1+\alpha, 4-\alpha)$	$\leq (36\alpha, 21-6\alpha)$
demand	$\approx (4+4\alpha, 16-4\alpha)$	$\geq (8+2\alpha, 14-2\alpha)$	$\leq (3+2\alpha, 9-2\alpha)$	

Put $\alpha = 0$ in the above fuzzy interval integer transportation problem. We get the following fuzzy interval integer transportation problem with variables $[x_{ij}^1, x_{ij}^4]$ for all i and j corresponding to the above interval integer transportation problem.

Table 3

	1	2	3	Supply
1	(1,4)	(2,11)	(2,8)	$\approx$(2,11)
2	(2,14)	(1,7)	(0,3)	$\geq$(3,12)
3	(4,16)	(3,21)	(1,4)	$\leq (3,21)$
demand	$\approx$(4,16)	$\geq$(8,14)	$\leq$(3,9)	

Now, the upper bound integer transportation problem of the above fuzzy interval integer transportation problem is as follow.

Table 4

	1	2	3	Supply
1	4	11	8	≈ 11
2	14	7	3	≥ 12
3	16	21	4	≤ 21
demand	≈ 16	≥ 14	≤ 9	

Convert the all inequalities into equalities; we get the following transportation problem

Table 5

	1	2	3	Supply
1	4	11	8	= 11
2	14	7	3	= 12
3	16	21	4	= 21
demand	= 16	= 14	= 9	

Now, using the zero method, the optimal solution to the upper bound integer transportation problem is

Table 6

	1	2	3	4	Supply
1	4 [6]	11	8	0[5]	= 11
2	14	7[3]	3[9]	0	= 12
3	16[10]	21[11]	4	0	= 21
demand	= 16	= 14	= 9	5	= 44

Now, using the step 8, we get the following solution for the upper bound integer transportation problem.

Table 7

	1	2	3	Supply
1	4 [11]	11	8	≈ 11
2	14	7[3]	3[9]	≥ 12
3	16[10]	21[11]	4	≤ 21
demand	≈ 16	≥ 14	≤ 9	

So, the optimal solution of upper bound integer transportation problem is $\overline{x}^4_{11} = 11, \overline{x}^4_{22} = 3, \overline{x}^4_{23} = 9, \overline{x}^4_{31} = 10, \overline{x}^4_{32} = 11$ and the transportation cost is $min.z_4 = 483$. Now, the lower bound integer transportation problem of the above fuzzy interval integer transportation problem is as follow.

Table 8

	1	2	3	Supply
1	1	2	2	≈ 2
2	2	1	0	≥ 3
3	4	3	1	≤ 3
demand	≈ 4	≥ 8	≤ 3	

Similarly using Zero method and Step 8, we get the optimal solution of lower bound integer transportation problem is

Table 9

	1	2	3	Supply
1	1 [4]	2	2	≈ 2
2	2	1[5]	0[3]	≥ 3
3	4	3[3]	1	≤ 3
demand	≈ 4	≥ 8	≤ 3	

$\overline{x}_{11}^1 = 4, \overline{x}_{22}^1 = 5, \overline{x}_{23}^1 = 3, \overline{x}_{32}^1 = 3$ and the transportation cost is $Min.z_1 = 18$.
Put $\alpha = 1$ in the above fuzzy interval integer transportation problem. We get the following fuzzy interval integer transportation problem with variables $[x_{ij}^2, x_{ij}^3]$ for all i and j.

Table 10

	1	2	3	Supply
1	(2,3)	(5,8)	(4,6)	$\approx$ (5, 8)
2	(6, 10)	(3,5)	(1,2)	$\geq$ (6,9)
3	(8,12)	(9,12)	(2,3)	$\leq$ (9,15)
demand	$\approx$ (8,12)	$\geq$ (10, 12)	$\leq$ (5, 7)	

Now, the upper bound integer transportation problem of the above fuzzy interval integer transportation problem is as follows.

Table 11

	1	2	3	Supply
1	3	8	6	≈ 8
2	10	5	2	≥ 9
3	12	15	3	≤ 15
demand	≈ 12	≥ 12	≤ 7	

Similarly, using Zero method and step 8, we get the following optimal solution for the upper bound integer transportation problem.

Table 12

	1	2	3	Supply
1	3[8]	8	6	≈ 8
2	10	5[2]	2[7]	≥ 9
3	12[5]	15[10]	3	≤ 15
demand	≈ 12	≥ 12	≤ 7	

So, the optimal solution of upper bound integer transportation problem is $\overline{\chi}_{11}^{3} = 8, \overline{\chi}_{22}^{3} = 2, \overline{\chi}_{23}^{3} = 7, \overline{\chi}_{31}^{3} = 5, \overline{\chi}_{32}^{3} = 10$ and the transportation cost is $min.z_3 = 258$.
Now, the lower bound integer transportation problem of the above fuzzy interval integer transportation problem is as follows.

Table 13

	1	2	3	Supply
1	2	5	4	≈ 5
2	6	3	1	≥ 6
3	8	9	2	≤ 9
demand	≈ 8	≥ 10	≤ 5	

Similarly using Zero method and Step 8, we get the optimal solution of lower bound integer transportation problem is

Table 14

	1	2	3	Supply
1	2[5]	5	4	≈ 5
2	6	3[4]	1[5]	≥ 6
3	8[3]	9[6]	2	≤ 9
demand	≈ 8	≥ 10	≤ 5	

$\overline{\chi}_{11}^{2} = 5, \overline{\chi}_{22}^{2} = 4, \overline{\chi}_{23}^{2} = 5, \overline{\chi}_{31}^{2} = 3, \overline{\chi}_{32}^{2} = 6$ and the transportation cost is $Min.z_2 = 105$.
Hence, the fuzzy optimal solution for the given fuzzy integer transportation problem is $\widetilde{\chi}_{11} \approx (4, 5, 8, 11), \widetilde{\chi}_{22} \approx (5, 4, 2, 3), \widetilde{\chi}_{23} \approx (3, 5, 7, 9), \widetilde{\chi}_{31} \approx (0, 3, 5, 10)$ and $\widetilde{\chi}_{32} \approx (3, 6, 10, 11)$ with the fuzzy objective value $\widetilde{z} = (18, 105, 258, 483)$.

8 Conclusion

We have attempted to develop the separation method based on zero method provides an optimal solution of the fuzzy interval integer transportation problem with mixed constraints. This method is a systematic procedure, which is very simple, easy to understand and apply. This method provides more options and can be served an important tool for the decision makers when they are handling various types of logistic problems interval parameters.

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I- LACUNARY VECTOR VALUED SEQUENCE SPACES IN $2-$NORMED SPACES VIA ORLICZ FUNCTION

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Abstract

In this article we introduce I- convergence of some lacunary vector valued sequences with respect to an Orlicz function in 2-normed spaces.

1 Introduction

The notion of ideal convergence was introduced first by P. Kostyrko et al [7] as a generalization of statistical convergence

The concept of $2-$normed spaces was initially introduced by Gähler [4] in the 1960's. Since then, this concept has been studied by many authors (see, for instance ([13],[11]).

Recently Savas ([14],[15]) defined some new sequence spaces by using Orlicz function and ideal convergence in 2-normed spaces.

In this article by using Orlicz functions and ideal convergence of sequences we introduce I- convergence of lacunary sequences with respect to an Orlicz function in 2-normed spaces.

Let $(X, \|.\|)$ be a normed space. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is said to be statistically convergent to $x \in X$ if the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ has natural density zero for each $\varepsilon > 0$.

A family $\mathcal{I} \subset 2^Y$ of subsets a nonempty set Y is said to be an ideal in Y if **(i)** $\emptyset \in \mathcal{I}$; **(ii)** $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$; **(iii)** $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ (see, [7],[8]) .

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}-$convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ belongs to $\mathcal{I}$ ([7, 8]).

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Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A $2-$norm on X is a function $\|.,.\| : X \times X \to R$ which satisfies (i) $\|x,y\| = 0$ if and only if x and y are linearly dependent; (ii) $\|x,y\| = \|y,x\|$; (iii) $\|\alpha x, y\| = |\alpha| \|x,y\|$, $\alpha \in \mathbb{R}$; (iv) $\|x, y+z\| \leq \|x,y\| + \|x,z\|$. The pair $(X, \|.,.\|)$ is then called a $2-$normed space [5]. As an example of a $2-$normed space we may take $X = \mathbb{R}^2$ being equipped with the $2-$norm $\|x,y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x_1, x_2\|_E = abs\left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \right)$$

Recall that $(X, \|.,.\|)$ is a $2-$Banach space if every Cauchy sequence in X is convergent to some x in X.

Recall in [9] that an Orlicz function $M : [0,\infty) \to [0,\infty)$ is continuous, convex, non-decreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

Subsequently Orlicz function was used to define sequence spaces by Parashar and B.Choudhary [10] and others. An Orlicz function M can always be represented in the following integral form: $M(x) = \int_0^x p(t)dt$ where p is the known kernel of M, right differential for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non-decreasing and $p(t) \to \infty$ as $t \to \infty$.

If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$ then this function is called Modulus function, which was presented and discussed by Ruckle [12] and Maddox [6].

An Orlicz function is said to satisfy Δ_2- condition if there exists a positive constant K such that $M(2x) \leq KM(x)$ for all $x \geq 0$.

Note that if M is an Orlicz function then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$.

2 Main Results

Let I be an admissible ideal, M be an Orlicz function, $(X, \|.,.\|)$ be a $2-$normed space and $p = (p_k)$ be a sequence of positive real numbers. By $S(2-X)$ we denote the space of all sequences defined over $(X, \|.,.\|)$. Now we define the following sequence spaces:

$$W^I(N_\theta, M, p, \|.,.\|) =$$

$$\left\{ x \in S(2-X) : \begin{array}{c} \left\{ r \in \mathbb{N} : h_r^{-1} \sum\limits_{k \in I_r} \left[M\left(\left\| \frac{x_k - L}{\rho}, z \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I \\ \text{for some } \rho > 0, \ L > 0 \text{ and each } z \in X \end{array} \right\},$$

$$W_0^I(N_\theta, M, p, \|.,.\|) =$$

$$\left\{ x \in S(2-X) : \begin{array}{c} \left\{ r \in \mathbb{N} : h_r^{-1} \sum\limits_{k \in I_r} \left[M \left\| \frac{x_k}{\rho}, z \right\| \right]^{p_k} \geq \varepsilon \right\} \in I \\ \text{for some } \rho > 0 \text{ and each } z \in X \end{array} \right\},$$

$$W_\infty (N_\theta M, \Delta^m, p, \|, ., \|) =$$

$$\left\{ x \in S(2-X) : \begin{array}{c} \exists K > 0 \text{ s.t} \sup\limits_{r \in \mathbb{N}} : h_r^{-1} \sum\limits_{k \in I_r} \left[M \left(\left\| \frac{x_k}{\rho}, z \right\| \right) \right]^{p_{k,l}} \leq K \\ \text{for some } \rho > 0, \text{ and each } z \in X \end{array} \right\},$$

$$W_\infty^I (N_\theta, M, p, \|, ., \|) =$$

$$\left\{ x \in S(2-X) : \begin{array}{c} \exists K > 0 \ni \left\{ r \in \mathbb{N} : h_r^{-1} \sum\limits_{k \in I_r} \left[M \left\| \frac{x_k}{\rho}, z \right\| \right]^{p_k} \geq K \right\} \in I \\ \text{for some } \rho > 0 \text{ and each } z \in X \end{array} \right\}.$$

The following well-known inequality will be used in the study.

$$0 \leq p_k \leq \sup p_k = H, \; D = \max\left(1, 2^{H-1}\right)$$

then

$$|a_k + b_k|^{p_k} \leq D \left\{ |a_k|^{p_k} + |b_k|^{p_k} \right\}$$

for all k and $a_k, b_k \in C$

Theorem 1 $W^I (N_\theta, M, p, \|, ., \|)$, $W_0^I (N_\theta, M, p, \|, ., \|)$, $W_\infty^I (N_\theta, M, p, \|, ., \|)$ *are linear spaces.*

Proof. We will prove the assertion for $W_0^I (N_\theta, M, p, \|, ., \|)$only and the others can be proved similarly. Assume that $x, y \in W_0^I (N_\theta, M, p, \|, ., \|)$ and $\alpha, \beta \in \mathbb{R}$. So

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left[M \left(\left\| \frac{x_k}{\rho_1}, z \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I \text{ for some } \rho_1 > 0$$

and

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left[M \left(\left\| \frac{x_k}{\rho_2}, z \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I \text{ for some } \rho_2 > 0.$$

Since $\|,.,\|$ is a 2−norm, and M is an Orlicz function the following inequality holds:

$$
\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\left\|\frac{(\alpha x_k+\beta y_k)}{(|\alpha|\rho_1+|\beta|\rho_2)},z\right\|\right)\right]^{p_k}\\
\leq\ & D\frac{1}{h_r}\sum_{k\in I_r}\left[\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}M\left(\left\|\frac{x_k}{\rho_1},z\right\|\right)\right]^{p_k}\\
+\ & D\frac{1}{h_r}\sum_{k\in I_r}\left[\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}M\left(\left\|\frac{y_k}{\rho_2},z\right\|\right)\right]^{p_k}\\
\leq\ & DF\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\left\|\frac{x_k}{\rho_1},z\right\|\right)\right]^{p_k}\\
+\ & DF\frac{1}{h_r}\sum_{k\in I_n}\left[M\left(\left\|\frac{y_k}{\rho_2},z\right\|\right)\right]^{p_k}
\end{aligned}
$$

where

$$
F=\max\left[1,\left(\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}\right)^H,\left(\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}\right)^H\right]
$$

From the above inequality we get

$$
\begin{aligned}
&\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\left\|\frac{(\alpha x_k+\beta y_k)}{(|\alpha|\rho_1+|\beta|\rho_2)},z\right\|\right)\right]^{p_k}\geq\varepsilon\right\}\\
\subseteq\ &\left\{r\in\mathbb{N}:DF\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\left\|\frac{x_k}{\rho_1},z\right\|\right)\right]^{p_k}\geq\frac{\varepsilon}{2}\right\}\\
\cup&\left\{r\in\mathbb{N}:DF\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\left\|\frac{y_k}{\rho_2},z\right\|\right)\right]^{p_k}\geq\frac{\varepsilon}{2}\right\}.
\end{aligned}
$$

Two sets on the right hand side belong to I and this completes the proof. It is also easy to verify that the space $W_\infty(N_\theta,M,p,\|,.,\|)$ is also a linear space and moreover we have

Theorem 2 *If M is an Orlicz function and (p_k) is bounded sequence of strictly positive real numbers then $W_\infty(N_\theta,M,p,\|,.,\|)$ is a paranormed space with respect to paranorm g defined by*

$$
g(x)=\sum_{k\in I_r}\|x_k,z\|+\inf\left\{\rho^{\frac{p_r}{H}}:\sup_k\left[M\left(\left\|\frac{x_k}{\rho},z\right\|\right)\right]^{p_k}\right.
$$

$$
\left.\leq 1,\ \rho>0,\ r=1,2,...\right\},\ \textit{each } z\in X
$$

Proof. That $g_n(\theta)=0$ and $g_n(-x)=g(x)$ are easy to see. So we omit them. (iii) Let us take $x=(x_k)$ and $y=(y_k)$ in $W_\infty(N_\theta,M,p,\|,.,\|)$. Let

$$
A(x)=\left\{\rho>0:\sup_k\left[M\left(\left\|\frac{x_k}{\rho},z\right\|\right)\right]^{p_k}\leq 1,\forall z\in X,\right\},
$$

$$A(y)=\left\{\rho>0:\sup_k\left[M\left(\left\|\frac{y_k}{\rho},z\right\|\right)\right]^{p_k}\leq 1,\forall z\in X,\right\}.$$

Let $\rho_1\in A(x)$ and $\rho_2\in A(y)$. Then if $\rho=\rho_1+\rho_2$, then we have

$$\begin{aligned}M\left(\left\|\frac{(x_k+y_k)}{\rho},z\right\|\right) &\leq \frac{\rho_1}{\rho_1+\rho_2}M\left(\left\|\frac{x_k}{\rho_1},z\right\|\right)\\ &\quad+\frac{\rho_2}{\rho_1+\rho_2}M\left(\left\|\frac{y_k}{\rho_2},z\right\|\right).\end{aligned}$$

Thus $\sup\limits_k M\left(\left\|\frac{(x_k+y_k)}{\rho_1+\rho_2},z\right\|\right)^{p_k}\leq 1$ and

$$\begin{aligned}g_n(x+y) &\leq \inf_{z\in X}\sum_{k\in I_r}\|x_k+y_k,z\|\\ &\quad+\inf\left\{(\rho_1+\rho_2)^{\frac{p_n}{H}}:\rho_1\in A(x),\rho_2\in A(y)\right\}\\ &\leq \inf_{z\in X}\sum_{k\in I_r}\|x_k,z\|+\inf\left\{\rho_1^{\frac{p_n}{H}}:\rho_1\in A(x)\right\}\\ &\quad+\inf_{z\in X}\sum_{k\in I_r}\|y_k,z\|+\inf\left\{\rho_2^{\frac{p_n}{H}}:\rho_2\in A(y)\right\}\\ &= g_n(x)+g_n(y).\end{aligned}$$

(iv) Let $\sigma^m\to\sigma$ where $\sigma,\sigma^m\in\mathbb{C}$ and let $g_n(x^m-x)\to 0$ as $m\to\infty$. We have to show that $g_n(\sigma^m x^m-\sigma x)\to 0$ as $m\to\infty$. Let

$$A(x^m)=\left\{\rho_m>0:\sup_k\left[M\left(\left\|\frac{x_k^m}{\rho_m},z\right\|\right)\right]^{p_k}\leq 1,\forall z\in X,\right\},$$

$$A(x^m-x)=\left\{\rho_m^{/}>0:\sup_k\left[M\left(\left\|\frac{(x_k^m-x_k)}{\rho_m^{/}},z\right\|\right)\right]^{p_k}\leq 1,\forall z\in X,\right\}.$$

If $\rho_m\in A(x^m)$ and $\rho_m^{/}\in A(x^m-x)$ then we observe that

$$M\left(\left\|\frac{(\sigma^m x_k^m-\sigma x_k)}{\rho_m|\sigma^m-\sigma|+\rho_m^{/}|\sigma|},z\right\|\right)$$

$$\begin{aligned}&\leq M\left(\left\|\frac{(\sigma^m x_k^m-\sigma x_k^m)}{\rho_m|\sigma^m-\sigma|+\rho_m^{/}|\sigma|},z\right\|+\left\|\frac{(\sigma x_k^m-\sigma x_k)}{\rho_m|\sigma^m-\sigma|+\rho_m^{/}|\sigma|},z\right\|\right)\\ &\leq \frac{|\sigma^m-\sigma|\,\rho_m}{\rho_m|\sigma^m-\sigma|+\rho_m^{/}|\sigma|}M\left(\left\|\frac{(x_k^m)}{\rho_m},z\right\|\right)\\ &\quad+\frac{|\sigma|\,\rho_m^{/}}{\rho_m|\sigma^m-\sigma|+\rho_m^{/}|\sigma|}M\left(\left\|\frac{(x_k^m-x_k)}{\rho_m^{/}},z\right\|\right).\end{aligned}$$

From the above inequality it now readily follows that

$$\left(M\left(\left\|\frac{(\sigma^m x_k^m-\sigma x_k)}{\rho_m|\sigma^m-\sigma|+\rho_m^{/}|\sigma|},z\right\|\right)\right)^{p_k}\leq 1$$

and consequently

$$g_n(\sigma^m x^m - \sigma x) \leq \inf_{z\in X} \sum_{k\in I_n} \|\sigma^m x_k^m - \sigma x_k, z\|$$

$$+\inf\left\{\left(\rho_m |\sigma^m - \sigma| + \rho_m^{/} |\sigma|\right)^{\frac{p_n}{H}} : \rho_m \in A(x^m), \rho_m^{/} \in A(x^m - x)\right\}$$

$$\leq |\sigma^m - \sigma| \inf_{z\in X} \sum_{k\in I_n} \|x_k^m, z\| + |\sigma| \inf_{z\in X} \sum_{k\in I_n} \|x_k^m - x_k, z\|$$

$$+ (|\sigma^m - \sigma|)^{\frac{p_n}{H}} \inf\left\{\rho_m^{\frac{p_n}{H}} : \rho_m \in A(x^m)\right\}$$

$$+(|\sigma|)^{\frac{p_n}{H}} \inf\left\{(\rho_m^{/})^{\frac{p_n}{H}} : \rho_m^{/} \in A(x^m - x)\right\}$$

$$\leq \max\left\{|\sigma^m - \sigma|, (|\sigma^m - \sigma|)^{\frac{p_n}{H}}\right\} g_n(x^m)$$

$$+\max\left\{|\sigma|, (|\sigma|)^{\frac{p_n}{H}}\right\} g_n(x^m - x).$$

Note that $g_n(x^m) \leq g_n(x) + g_n(x^m - x)$ for all $m \in \mathbb{N}$. Hence by our assumption the right hand side tends to 0 as $m \to \infty$ and the result follows. This completes the proof of the theorem.

Corollary 1. If one considers the sequence space $W_\infty^I(N_\theta, M, p, \|,.,\|)$ which is larger than the space $W_\infty(N_\theta, M, p, \|,.,\|)$ the construction of the paranorm is not clear and we leave it as an open problem.

Theorem 3 *Let* M, M_1, M_2, *be Orlicz functions. Then we have* $W_0^I(N_\theta, M_1, p, \|,.,\|) \subseteq W_0^I(N_\theta, M \circ M_1, p, \|,.,\|)$ *provided* (p_k) *is such that* $H_0 = \inf p_k > 0$.

Proof. (i) For given $\varepsilon > 0$, first choose $\varepsilon_0 > 0$ such that $\max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < \varepsilon$. Now using the continuity of M choose $0 < \delta < 1$ such that $0 < t < \delta \Rightarrow M(t) < \varepsilon_0$. Let $(x_k) \in W_0(N_\theta, M_1, p, \|,.,\|)$. Now from the definition

$$A(\delta) = \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{n\in I_r} \left[M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} \geq \delta^H\right\} \in I.$$

Thus if $r \notin A(\delta)$ then

$$\frac{1}{h_r} \sum_{k\in I_r} \left[M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} < \delta^H$$

$$\text{i.e.} \sum_{n\in I_r} \left[M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} < h_r \delta^H$$

$$\text{i.e.} \left[M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} < \delta^H \text{ for all } k \in I_r$$

$$\text{i.e.} M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right) < \delta \text{ for all } k \in I_r.$$

Hence from above using the continuity of M we must have

$$M\left(M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right) < \varepsilon_0 \text{ for all } k \in I_r$$

which consequently implies that

$$\sum_{k \in I_r}\left[M\left(M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right)\right]^{p_k} < h_r \max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < h_r\varepsilon,$$

$$\text{i.e.} \frac{1}{h_r}\sum_{k \in I_r}\left[M\left(M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right)\right]^{p_k} < \varepsilon.$$

This shows that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r}\left[M\left(M_1\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right)\right]^{p_k} \geq \varepsilon\right\} \subset A(\delta)$$

and so belongs to I. This proves the result.

Theorem 4 *Let the sequence (p_k) be bounded, then $W_0^I(N_\theta, M, p, \|, ., \|) \subseteq W^I(N_\theta, M, p, \|, ., \|) \subseteq W_\infty^I(N_\theta, M, p, \|, ., \|)$.*

Proof. Let $x = (x_k) \in W_0^I(N_\theta, M, p, \|, ., \|)$. Then given $\varepsilon > 0$ we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r}\left[M\left(\left\|\frac{x_k}{\rho}, z\right\|\right)\right]^{p_k} \geq \varepsilon\right\} \in I \text{ for some } \rho > 0.$$

Since M is non-decreasing and convex it follows that

$$\frac{1}{h_r}\sum_{k \in I_r}\left[M\left(\left\|\frac{x_k}{2\rho}, z\right\|\right)\right]^{p_k}$$

$$\leq \frac{D}{h_r}\sum_{k \in I_r}\frac{1}{2^{p_k}}\left[M\left(\left\|\frac{x_k - x_0}{\rho}, z\right\|\right)\right]^{p_k} + \frac{D}{h_r}\sum_{k \in I_r}\frac{1}{2^{p_k}}\left[M\left(\left\|\frac{x_0}{\rho}, z\right\|\right)\right]^{p_k}$$

$$\leq \frac{D}{h_r}\sum_{k \in I_r}\left[M\left(\left\|\frac{x_k - x_0}{\rho}, z\right\|\right)\right]^{p_k} + Dmax\left\{1, sup\left[M\left(\left\|\frac{x_0}{\rho}, z\right\|\right)\right]^{p_k}\right\}.$$

Hence we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r}\left[M\left(\left\|\frac{x_k}{2\rho}, z\right\|\right)\right]^{p_k} \geq \varepsilon\right\}$$

$$\subseteq \left\{r \in \mathbb{N} : \frac{D}{h_r}\sum_{k \in I_r}\left[M\left(\left\|\frac{x_{k+1}}{\rho}, z\right\|\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\}$$

$$\cup \left\{r \in \mathbb{N} : max\left\{1, sup\left[M\left(\left\|\frac{x_0}{\rho}, z\right\|\right)\right]^{p_k}\right\} \geq \frac{\varepsilon}{2}\right\}.$$

Since the set on the right hand side belongs to I so does the left hand side. The inclusion $W^I(N_\theta, M, p, \|,.,\|) \subseteq W^I_\infty(N_\theta, M, p, \|,.,\|)$ is obvious.

Theorem 5 *1. Let* $0 < \inf p_k \leq p_k < 1$. *Then*

$$W^I(N_\theta, M, p, \|,.,\|) \subseteq W^I(N_\theta, M, \|,.,\|).$$

2. Let $1 \leq p_k \leq \sup p_k < \infty$. *Then*

$$W^I(N_\theta, M, \|,.,\|) \subseteq W^I(N_\theta, M, p, \|,.,\|).$$

Proof. Let $x \in W^I(N_\theta, M, p, \|,.,\|)$, since $0 < \inf p_{k,l} \leq 1$, we obtain the following:

$$\left\{r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} \left[M\left(\left\|\tfrac{x_k - L}{\rho}, z\right\|\right)\right] \geq \varepsilon\right\}$$

$$\subseteq \left\{r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} \left[M\left(\left\|\tfrac{x_k - L}{\rho}, z\right\|\right)\right]^{p_{k,l}} \geq \varepsilon\right\} \in I.$$

Thus $x \in W^I(N_\theta, M, \|,.,\|)$. Let us establish part (2). Let $p_k > 1$ for each k , and $\sup_k p_k < \infty$. Let $x \in W^I(N_\theta, M, \|,.,\|)$. Then for each $0 < \epsilon < 1$ there exists a positive integer N such that

$$h_r^{-1} \sum_{k \in I_r} \left[M\left(\left\|\frac{x_k - L}{\rho}, z\right\|\right)\right] \leq \epsilon < 1$$

for all $r \geq N$. This implies that

$$\left\{r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} \left[M\left(\left\|\tfrac{x_k - L}{\rho}, z\right\|\right)\right]^{p_{k,l}} \geq \varepsilon\right\}$$

$$\subseteq \left\{r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} \left[M\left(\left\|\tfrac{x_k - L}{\rho}, z\right\|\right)\right] \geq \varepsilon\right\} \in I.$$

Therefore $x \in W^I(N_\theta, M, p, \|,.,\|)$. This completes the proof.

Definition 1 *Let X be a sequence space. Then X is called solid if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in N$.*

We now have

Theorem 6 *The sequence spaces $W^I_0(N_\theta, M, p, \|,.,\|)$, $W^I_\infty(N_\theta, M, p, \|,.,\|)$ are solid.*

Proof. We give the proof for $W^I_0(N_\theta, M, p, \|,.,\|)$. Let $(x_k) \in W^I_0(N_\theta, M, p, \|,.,\|)$ and (α_k) be sequences of scalars such that $|\alpha_k| \leq 1$ for all $k \in N$. Then we have

$$\left\{r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} \left[\left(M \left\|\frac{(\alpha_k x_k)}{\rho}, z\right\|\right)\right]^{p_k} \geq \varepsilon\right\}$$

$$\subseteq \left\{r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} \left[\left(M \left\|\frac{x_k)}{\rho}, z\right\|\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

Hence $(\alpha_k x_k) \in W_0^I(N_\theta, M, p, \|., .\|)$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in N$ whenever $(x_k) \in W_0^I(N_\theta, M, p, \|., .\|)$.

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Bianchi Type-I Cosmological Model Filled with Viscous Fluid in a Modified Brans-Dicke Cosmology

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Abstract

Adding the cosmological term, which is assumed to be variable in Brans-Dicke theory we have discussed about a Bianchi type-I cosmological model filled with viscous fluid with free gravitational field of Petrov type-D. The effect of viscousity on various kinematical parameters has been discussed. Finally, this model has been transformed to the original form (1961) of Brans-Dicke theory (including a variable cosmological term).

1 Introduction

After the cosmological constant was first introduced into general relativity by Einstein, its significance was studied by various cosmologists (for example [1]), but no satisfactory results of its meaning have been reported as yet. Zel' dovich [2] has tried to visualize the meaning of this term from the theory of elementary particles. Further, Linde [3] has argued that the cosmological term arises from spontaneous symmetry breaking and suggested that the term is not a constant but a function of temperature. Also Drietlein [4] connects the mass of Higg's scalar boson with both the cosmological term and the gravitational constant. In cosmology the term may be understood by incorporation with Mach's principle, which

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suggests the acceptance of Brans-Dicke Lagrangian as a realistic case [5]. The investigation of particle physics within the context of the Brans-Dicke Lagrangian [6] has stimulated the study of the cosmological term with a modified Brans-Dicke Lagrangian in cosmology and elementary particle physics. Endo and Fukui [7] have studied the variable cosmological term in Brans-Dicke [5] and elementary particle physics (specially in the context of Dirac's large number hypothesis [8], [9]).

Further, astronomical observations of the large scale distribution of galaxies in our universe have shown that the distribution of matter can be satisfactorily discribed by a perfect fluid. It has, however, been conjectured that some time during an earlier phase in the evolution of the universe when galaxies were formed, the material distribution behaved like a viscous fluid ([10], [p. 124]). It is therefore of interest to obtain cosmological models for such distributions. It is also well known that there is a certain degree of anisotropy in the actual universe. Therefore, we have choosen the metric for the cosmological model to be Bianchi type-I. Thus, in this paper we have considered a Bianchi type-I cosmological model filled with viscous fluid in a modified Brans-Dicke theory in which the variable cosmological term Q is an explicit function of a scalar field ϕ as proposed by Bergmann [11] and Wagoner [12] and discussed in detail by Endo and Fukui [7].

The Brans-Dicke field equations with cosmological term Q [7] are :

$$G_{ij} + g_{ij}Q = \frac{8\pi}{\phi}T_{ij} + \frac{\omega}{\phi^2}\left(\phi_i\phi_j - \frac{1}{2}g_{ij}\phi_{,k}\,\phi^{,k}\right)$$

$$+\frac{1}{\phi}(\phi_i;j - g_{ij}\Box\phi) \tag{1.1}$$

$$\Box\phi == \frac{8\pi\mu T}{(2\omega+3)} \tag{1.2}$$

$$Q = \frac{(2\omega+3)}{4}\frac{(1-\mu)}{\mu}\frac{\Box\phi}{\phi} = \frac{8\pi(1-\mu)}{4\phi}T \tag{1.3}$$

where the constant μ shows how much our theory including $Q(\phi)$ deviates from that of Brans and Dicke and as usual ω is coupling constant and T_{ij} is energy-momentum tensor for a viscous fluid distribution [13]. Semicolons denote covariant differentiation with respect to the metric g_{ij} and commas mean partial differentiation with respect to the coordinate x^i. The theory can also be represented in a different form [14] under a unit transformation (UT) in which length, time and reciprocal mass are scaled by the function $\lambda^{\frac{1}{2}}(x)$. Then under the conformal transformation :

$$g_{ij} \to \overline{g_{ij}} = \phi g_{ij} \tag{1.4}$$

the equation (1.1) - (1.3) go to the form

$$\overline{G_{ij}} + \overline{g_{ij}}\overline{Q} = 8\pi\overline{T}_{ij} + \frac{1}{2}(2\omega+3)\left(\wedge_i\wedge_j - \frac{1}{2}\overline{g}_{ij}\wedge_k\wedge^k\right) \tag{1.5}$$

$$\overline{\Box\wedge} = \frac{8\pi\mu\overline{T}}{(2\omega+3)}, \wedge = \log\phi \tag{1.6}$$

$$\overline{Q} = \frac{(2\omega+3)}{4}\frac{(1-\mu)}{\mu}\overline{\square}\Lambda = \frac{8\pi(1-\mu)}{4}\overline{T} \tag{1.7}$$

where the barred quantities are defined in terms of as their unbarred counterparts are defined in terms of the unbarred metric g_{ij} and all barred operations are performed with respect to the barred metric and barred Christoffel symbols. In section-2 the Bianchi type-I metric is considered and the energy-momentum tensor is taken to be that of a viscous fluid [3]. In section-3 we have obtained pressure, density expressions for spatially homogeneous and anisotropic Bianchi type-I cosmological model which is also of petrov type-D. The effect of viscousity on various kinematical parameters has been also discussed. It is found that the kinematic viscousity prevents shear, expansion and the free gravitational field from withering away. Finally in section-4 we have transformed this model to the 1961 form of Brans-Dicke theory.

2 Derivation Of The Line-Element

We use here the spatially homogeneous and anisotropic Bianchi type-I line element in the form :

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + C^2dz^2 \tag{2.1}$$

where the quantities A, B and C are functions of t only. The energy-momentum tensor for a viscous fluid distribution is given by (Landau and Lifshitz [13])

$$\overline{T}i^k = (\overline{\in} + \overline{p})\overline{v}i\overline{v}^k + \overline{p}\,\overline{\in}^k i - \eta(\overline{v}^k i; + \overline{v}^k; i + \overline{v}^k\overline{v}^1\overline{v}i; 1$$

$$+ \overline{v}i\overline{v}^1\overline{v}^k; 1) - (\xi - \frac{2}{3}\eta)\overline{v}^1; 1(\overline{g}^k i + \overline{v}i\overline{v}^k) \tag{2.2}$$

together with

$$\overline{g}_{ij}\overline{v}^i\overline{v}^j = -1 \tag{2.3}$$

where being the isotropic pressure, the density, η and the two coefficients of viscousity and semicolons indicate covariant differentiation. v^i is the flow vector satisfying equation (2.3). We assume the coordinates to be comoving so that . Scalar field is also taken to be a function of t only. The field equations (1.5) and (1.6) for the line-element (2.1) are as follows

$$\left[\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4C_4}{BC}\right] + \overline{Q}$$

$$= 8\pi\left\{\overline{p} - 2\eta\left(\frac{A_4}{A}\right) - \left(\xi - \frac{2}{3}\eta\right)v^1_{;1}\right\} + \frac{(2\omega+3)}{4}\Lambda_4^2 \tag{2.4}$$

$$\left[\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4C_4}{AC}\right] + \overline{Q}$$

$$= 8\pi\left\{\overline{p} - 2\eta\left(\frac{B_4}{B}\right) - \left(\xi - \frac{2}{3}\eta\right)v^1_{;1}\right\} + \frac{(2\omega+3)}{4}\Lambda_4^2 \tag{2.5}$$

$$\left[\frac{A_{44}}{A}+\frac{B_{44}}{B}+\frac{A_4B_4}{AB}\right]+\overline{Q}$$

$$=8\pi\left\{\overline{p}-2\eta\left(\frac{C_4}{C}\right)-\left(\xi-\frac{2}{3}\eta\right)v^1_{;1}\right\}+\frac{(2\omega+3)}{4}\wedge_4^2 \qquad (2.6)$$

$$\left[\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}+\frac{B_4C_4}{BC}\right]+\overline{Q}=-8\pi\overline{\in}-\frac{(2\omega+3)}{4}\wedge_4^2 \qquad (2.7)$$

$$\wedge_{44}+\wedge_4\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)$$

$$=-\frac{8\pi\mu}{(2\omega+3)}\left[(3\overline{p}-\overline{\in})-3\xi\frac{d}{dt}\log(ABC)\right] \qquad (2.8)$$

The suffix 4 after the symbols A, B, C denotes ordinary differentiation with respect to t. Equation (2.4) - (2.8) are five equations in six unknowns A, B, C . The coefficients of viscousity are taken as constants. For complete determinacy of the system one extra condition is needed. One way is to impose an equation of state. The other alternative is a mathematical assumption on the space-time and then to discuss the physical nature of the universe. Although the distribution of matter at each point determines the nature of expansion in the model, the later is also affected by the free gravitational field through its effect on the expansion, vorticity and shear in the fluid flow. A prescription of such a field may therefore be made on a priori basis. The cosmological models of Robertson and Walker, as well as the universes of Einstein and De Sitter, have vanishing free gravitational fields. In this paper, we choose the free-gravitational field to be type-D which is of the next hierarchy of Petrov classification. This requires that, either

$(a) C^{12}_{12}=C^{13}_{13}$
Or
$(b) C^{12}_{12}=C^{23}_{23}$

Conditions (a) and (b) are identically satisfied if $B = C$ and $A = C$ respectively. However, we shall assume A, B, C to be unequal on account of the supposed anisotropy. In this paper we shall confine ourselves to the condition (a). The condition leads to

$$\frac{B_{44}}{B}-\frac{C_{44}}{C}+2\frac{A_4}{A}\left(\frac{C_4}{C}-\frac{B_4}{B}\right)=0 \qquad (2.9)$$

Subtracting equation (2.5) from equation (2.4), we get

$$\frac{B_{44}}{B}-\frac{A_{44}}{A}+\frac{B_4C_4}{BC}-\frac{A_4C_4}{AC}=16\pi\eta\left(\frac{B_4}{B}-\frac{A_4}{A}\right) \qquad (2.10)$$

Also, subtracting equation (2.6) from equation (2.5), we get

$$\frac{C_{44}}{C}-\frac{B_{44}}{B}+\frac{A_4C_4}{AC}-\frac{A_4B_4}{AB}=16\pi\eta\left(\frac{C_4}{C}-\frac{B_4}{B}\right) \qquad (2.11)$$

From equations (2.9) and (2.11), we have

$$3\frac{A_4}{A}\left(\frac{C_4}{C}-\frac{B_4}{B}\right)=16\pi\eta\left(\frac{C_4}{C}-\frac{B_4}{B}\right) \tag{2.12}$$

Since equation (2.12) gives

$$\frac{A_4}{A}=\frac{16\pi\eta}{3} \tag{2.13}$$

which on integration gives

$$A=Me^{\frac{16\pi\eta t}{3}} \tag{2.14}$$

where M being a constant of integration.

From (2.9) and (2.13) we get

$$\frac{B_{44}}{B}-\frac{C_{44}}{C}+\frac{32\pi\eta}{3}+\left(\frac{C_4}{C}-\frac{B_4}{B}\right)=0$$

which on integration gives

$$B_4C-BC_4=e^{\left(\frac{32\pi\eta t}{3}+b\right)} \tag{2.15}$$

where b being a constant of integration.

From equations (2.10) and (2.13) we get

$$\frac{B_{44}}{B}+\frac{B_4C_4}{BC}-16\pi\eta\left(\frac{B_4}{B}\right)-\frac{16\pi\eta}{3}\left(\frac{C_4}{C}\right)=\frac{512}{9}\pi^2\eta^2 \tag{2.16}$$

On substituting $\frac{B}{C}=\alpha, BC=\beta$ so that $B^2=\alpha\beta, C^2=\frac{\beta}{\alpha}$

Equation (2.15) reduces to

$$\left(\frac{\alpha_4}{\alpha}\right)\beta=e^{\left(\frac{32\pi\eta t}{3}+b\right)} \tag{2.17}$$

From equation (2.16) we have

$$9\beta_{44}-192\pi\eta\beta_4+1024\pi^2\eta^2\beta=0$$

After solving this equation gives

$$\beta=(k_1t+k_2)e^{\frac{32\pi\eta t}{3}} \tag{2.18}$$

where k_1 and k_2 are constants of integration.

From (2.17) and (2.18) we get

$$\alpha=(k_1t+k_2)^a \tag{2.19}$$

where a is another constant of integration.

From equations (2.18) and (2.19) we get

$$B^2 = \alpha\beta = (k_1 t + k_2)^{(1-a)} e^{32\pi\eta t} 3 \tag{2.20}$$

and

$$C^2 = \frac{\beta}{\alpha} = (k_1 t + k_2)^{(1-a)} e^{32\pi\eta t} 3 \tag{2.21}$$

Consequently the line-element takes the form

$$ds^2 = -dt^2 + M^2 e^{\frac{32\pi\eta t}{3}} dx^2 + (k_1 t + k_2)^{(1+a)} e^{\frac{32\pi\eta t}{3}} dy^2$$
$$+(k_1 t + k_2)^{(1+a)} e^{\frac{32\pi\eta t}{3}} dz^2 \tag{2.22}$$

By the following transformation of coordinates $Mx \to x, (k_1 t + k_2) \to t, z \to z$ we get

$$ds^2 = -dt^2 + e^{\frac{32\pi\eta t}{3}} dx^2 + t^{(1+a)} e^{\frac{32\pi\eta t}{3}} dy^2 + t^{(1-a)} e^{\frac{32\pi\eta t}{3}} dz^2 \tag{2.23}$$

3 Some Physical And Geometrical Features

The pressure and density in the model (2.23) are given by

$$8\pi\overline{p} = \frac{(a^2-1)}{4t^2} \sec^2 \left\{ \sqrt{\frac{(1-a^2)}{4(2\omega+3)}} \log(kt) \right\}$$
$$+8\pi\xi(t + 16\pi\eta) - \frac{16\pi\eta}{3t}(t^2 - 3) + \frac{768}{9}\pi^2\eta^2 + \overline{Q} \tag{3.1}$$

$$8\pi \in = \frac{(a^2-1)}{4t^2} \sec^2 \left\{ \sqrt{\frac{(1-a^2)}{4(2\omega+3)}} \log(kt) \right\}$$
$$-\frac{32\pi\eta}{3t} - \frac{768}{9}\pi^2\eta^2 - Q \tag{3.2}$$

Also the scalar field $\wedge$ is given by

$$\wedge = \log \sec^2 \left\{ \sqrt{\frac{(1-a^2)}{4(2\omega+3)}} \log(kt) \right\} \tag{3.3}$$

and

$$\overline{Q} = \frac{(1-\mu)}{4\mu} \left[\frac{(a^2-1)}{2t^2} \sec^2 \left\{ \sqrt{\frac{(1-a^2)}{4(2\omega+3)}} \log(kt) \right\} \right.$$

$$+\frac{32\pi\eta}{3t}+24\pi\xi(t+16\pi\eta)+\frac{3072}{9}\pi^2\eta^2\Bigg] \tag{3.4}$$

The model is real and the conditions hold when

$$a^2>1,\ \omega<-\frac{3}{2},\overline{Q}>0(\text{ i.e. } \mu<1) \tag{3.5}$$

The non-vanishing components of the Weyl's conformal curvature tensor C_{jk}^{jk} are :

$$\begin{aligned}
C_{14}^{14}=C_{23}^{23}&=\frac{1}{6}\left[\frac{(a^2-1)}{t^2}-\frac{512}{9}\pi^2\eta^2\right]e^{-\frac{32\pi\eta t}{3}}\\
C_{12}^{12}=C_{34}^{34}&=-\frac{1}{12}\left[\frac{(a^2-1)}{t^2}-\frac{512}{9}\pi^2\eta^2\right]e^{-\frac{32\pi\eta t}{3}}\\
C_{13}^{13}=C_{24}^{24}&=-\frac{1}{12}\left[\frac{(a^2-1)}{t^2}-\frac{512}{9}\pi^2\eta^2\right]e^{-\frac{32\pi\eta t}{3}}
\end{aligned} \tag{3.6}$$

Thus,

$$\begin{aligned}
-\frac{1}{2}C_{14}^{14}&=-\frac{1}{2}C_{23}^{23}=C_{12}^{12}=C_{34}^{34}=C_{13}^{13}=C_{24}^{24}\\
&=-\frac{1}{12}\left[\frac{(a^2-1)}{t^2}-\frac{512}{9}\pi^2\eta^2\right]e^{-\frac{32\pi\eta t}{3}}
\end{aligned}$$

The flow vector is given by

$$v^1=v^2=v^3=0,v^4=1 \tag{3.7}$$

It satisfies, so that the flow is geodetic.

Also $W_{ij}=0$.

The scalar of expansion is

$$\theta=\frac{(6\pi\eta t+1)}{3t} \tag{3.8}$$

The non-zero components of shear tensor σ_{ij} are

$$\begin{aligned}
\sigma_{11}&=\frac{(10\pi\eta t-1)}{3t}e^{\frac{32\pi\eta t}{3}}\\
\sigma_{22}&=\frac{(20\pi\eta t+3a+1)}{6t}t^{1+a}e^{\frac{32\pi\eta t}{3}}\\
\sigma_{33}&=\frac{(20\pi\eta t-3a+1)}{6t}t^{1-a}e^{\frac{32\pi\eta t}{3}}
\end{aligned} \tag{3.9}$$

$$\sigma_{44} = \frac{2(5\pi\eta t + 1)}{3t}$$

and the shear σ is

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{72t^2}\{1416\pi^2\eta^2t^2 + 352\pi\eta t + 18a^2 + 22\} \tag{3.10}$$

Thus the viscousity prevents the free gravitational field as well as the shear from withering away. It is also clear from equation (3.8) that the effect of viscousity is to retard expansion of the model.

The pressure, density, scalar field and cosmological constant are singular at

$$t = \left(\frac{1}{k}\right) \exp . \left\{\pi\sqrt{\frac{(2\omega+3)}{(1-a^2)}}\right\} \tag{3.11}$$

The model exists for a finite time

$$\left(\frac{1}{k}\right) \leq t < \left(\frac{1}{k}\right) \exp . \left\{\pi\sqrt{\frac{(2\omega+3)}{(1-a^2)}}\right\} \tag{3.12}$$

When $\mu = 1$, the cosmological term vanishes and the model (2.23) reduces into a Brans-Dicke analogue of one of the viscous model in general relativity.

4 Transformations Of The Solutions And Discussion

Under the transformation

$$\left.\begin{array}{ll} \overline{g}_{ij} \to g_{ij} = \frac{1}{\phi}\overline{g}_{ij}, & \overline{T}_{ij} \to T_{ij} = \phi\overline{T}_{ij} \\ \overline{T} \to T = \phi^2\overline{T}, & \overline{p} \to p = \phi^2\overline{p} \\ \overline{\in} \to \in = \phi^2\overline{\in}, & \phi \to \phi = e^{\wedge} \\ \overline{Q} \to Q = \phi\overline{Q}, & \overline{v^i} \to v^i = \phi^{\frac{1}{2}}\overline{v^i} \end{array}\right\} \tag{4.1}$$

the field equations (1.5) - (1.7) are changed into (1.1) - (1.3).

We now apply these transformations to the solutions obtained in section-3.

$$\phi = \sec^2\left\{\sqrt{\frac{(1-a^2)}{4(2\omega+3)}}\log(kt)\right\} \tag{4.2a}$$

$$g_{ij} = \sec^2\left\{\sqrt{\frac{(1-a^2)}{4(2\omega+3)}}\log(kt)\right\}\overline{g}_{ij} \tag{4.2b}$$

$$g_{11} = \sec^2 \left\{ \sqrt{\frac{(1-a^2)}{4(2\omega+3)}} \log(kt) \right\} e^{\frac{32\pi\eta t}{3}}$$

i.e.

$$g_{22} = \sec^2 \left\{ \sqrt{\frac{(1-a^2)}{4(2\omega+3)}} \log(kt) \right\} t^{(1+a)} e^{\frac{32\pi\eta t}{3}}$$

$$g_{33} = \sec^2 \left\{ \sqrt{\frac{(1-a^2)}{4(2\omega+3)}} \log(kt) \right\} t^{(1-a)} e^{\frac{32\pi\eta t}{3}}$$

$$g_{44} = -\sec^2 \left\{ \sqrt{\frac{(1-a^2)}{4(2\omega+3)}} \log(kt) \right\}$$

$$\begin{aligned} 8\pi p = {} & \frac{(1+\mu)}{2\mu} \frac{(a^2-1)}{4t^2} \sec^6 \left\{ \sqrt{\frac{(1-a^2}{4(2\omega+3)}} \log(kt) \right\} \\ & + \sec^4 \left\{ \sqrt{\frac{(1-a^2}{4(2\omega+3)}} \log(kt) \right\} \\ & \left[8\pi\xi(t+16\pi\eta) \frac{(\mu+3)}{4\mu} - \frac{16\pi\eta}{3t}(t^2-3) \right. \\ & \left. + \frac{8\pi\eta}{3t} \frac{(1-\mu)}{\mu} + \frac{768}{9\mu}\pi^2\eta^2 \right] \end{aligned} \tag{4.2c}$$

$$\begin{aligned} 8\pi \in = {} & \frac{(1+\mu)}{2\mu} \frac{(a^2-1)}{4t^2} \sec^6 \left\{ \sqrt{\frac{(1-a^2}{4(2\omega+3)}} \log(kt) \right\} \\ & - \sec^4 \left\{ \sqrt{\frac{(1-a^2)}{4(2\omega+3)}} \log(kt) \right\} \left[6\pi\xi(t+16\pi\eta) \frac{(1-\mu)}{\mu} \right. \\ & \left. + \frac{8\pi\eta}{3t} \left(\frac{3\mu+1}{\mu} + \frac{786}{9\mu} \right) + \frac{768}{9\mu}\pi^2\eta^2 \right] \end{aligned} \tag{4.2d}$$

$$\begin{aligned} Q = {} & \frac{(1-\mu)}{4\mu} \sec^2 \left\{ \sqrt{\frac{(1-a^2)}{4(2\omega+3)}} \log(kt) \right\} \\ & \left[\frac{(a^2-1)}{2t^2} \sec^2 \left\{ \sqrt{\frac{(1-a^2)}{4(2\omega+3)}} \log(kt) \right\} + \frac{32\pi\eta}{3t} \right. \\ & \left. + 24\pi\xi(t+16\pi\eta) + \frac{3072}{9}\pi^2\eta^2 \right] \end{aligned} \tag{4.2e}$$

$$v^1 = v^2 = v^3 = 0, v^4 = \sec \left\{ \sqrt{\frac{(1-a^2)}{4(2\omega+3)}} \log(kt) \right\} \tag{4.2f}$$

The reality conditions should also be imposed on the solutions in (4.2) similar to those in section 3. Model obtained in this paper is new and like other models with $p = \rho$ they may be used in the relativistic cosmology for the description of very early stages of the universe expansion.

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An Anisotropic Homogeneous Bianchi Type-I Cosmological Model in Self-Creation Cosmology

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Abstract

The paper presents an exact solution of spatially homogeneous and anisotropic Bianchi type-I cosmological model in Barber's second self-creation theory of gravitation which is of Petrov type-D. Some physical properties of this model are also discussed.

1 Introduction

Barber [1] proposed two self-creation cosmologies by modifying the Brans and Dicke [2] theory of gravitation and general theory of relativity. These modified theories create the universe out of self contained gravitational and matter fields. After that Brans [3] has pointed out that Barber's first theory is not only in disagreement with experiment, but is actually inconsistent. Barber's second theory is a modification of general relativity to a variable G-theory. In this theory the scalar field does not directly gravitate, but simply divides the matter tensor, acting as a reciprocal gravitational constant. It is postulated that this scalar field couples to the trace of the energy-momentum tensor. Hence, the field equations in Barber's second theory are

$$R_{ij} - \frac{1}{2}g_{ij}R = -8\pi\phi^{-1}T_{ij} \tag{1.1}$$

Bianchi Type-I, Homogeneous, Barber's Second Self-Creation Theory, Petrov Type-D, Cosmological Model.

AMS Subject Classification : 83D05, 83F05.

and

$$\Box\phi = \frac{8\pi}{3}\lambda T \tag{1.2}$$

where λ is a coupling constant to be determined from experiments. The measurements of the deflection of light restricts the value of the coupling to $|\lambda| < 10^{-1}$. In the limit $\lambda \to 0$ this theory approaches the standard general relativity theory in every respect. Barber [1] and Soleng [4] have discussed the F-R-W models while Reddy and Venkateswarlu [5] have studied the Bianchi type VI_o cosmological model in Barber's second theory of gravitation.

In this paper we have discussed about spatially homogeneous and anisotropic Bianchi type-I cosmological model in Barber's second self-creation theory of gravitation which is of Petrov type-D. Some physical properties of this model have been also discussed.

2 The Field Equations In Self-Creation Cosmology

We use here the spatially homogeneous and anisotropic Bianchi type-I line-element in the form

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + C^2dz^2 \tag{2.1}$$

where the quantities A, B and C are functions of t only.

The energy-momentum tensor T_ij for perfect fluid distribution is given by

$$T_{ij} = (\rho + p)V_iV_j + pg_{ij} \tag{2.2}$$

together with

$$g_{ij}V^iV^j = -1 \tag{2.3}$$

where p and ρ are proper pressure and energy density respectively and V^i are the components of the fluid four velocity. We assume the coordinates to be commoving so that $V^1 = V^2 = V^3 = 0$ and $V^4 = 1$. Scalar field ϕ is also a function of t only. The field equations (1.1) and (1.2) for the metric (2.1) can be written as

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4C_4}{BC} = -\frac{8\pi}{\phi}p \tag{2.4}$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4C_4}{AC} = -\frac{8\pi}{\phi}p \tag{2.5}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB} = -\frac{8\pi}{\phi}p \tag{2.6}$$

$$\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} + \frac{B_4C_4}{BC} = \frac{8\pi}{\phi}\rho \tag{2.7}$$

$$\phi_{44} + \phi_4\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) = \frac{8\pi}{3}\lambda(3p - \rho) \tag{2.8}$$

The suffix 4 after A, B, C and ϕ denotes ordinary differentiation with respect to t. Equations (2.4) - (2.8) are five equations in six unknowns A, B, C, p, ρ and ϕ. For complete determinancy of the system one extra condition is needed. One way is to impose an equation of state. The other alternative is a mathematical assumption on the space time and then to discuss the physical nature of the universe. We shall confine to the latter method in this paper and assume that $C^{12}_{12} = C^{13}_{13}$. The resulting space-time will obviously be of Petrov type-D. Thus, we have

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} + \frac{2A_4}{A}\left(\frac{C_4}{C} - \frac{B_4}{B}\right) = 0 \tag{2.9}$$

From (2.4) and (2.5) we get

$$\frac{B_{44}}{B} - \frac{A_{44}}{A} + \frac{B_4C_4}{BC} - \frac{A_4C_4}{AC} = 0 \tag{2.10}$$

Subtracting equation (2.6) from equation (2.5) we get

$$\frac{C_{44}}{C} - \frac{B_{44}}{B} + \frac{A_4C_4}{AC} - \frac{A_4B_4}{AB} = 0 \tag{2.11}$$

From (2.9) and (2.11), we have

$$\frac{3A_4}{A}\left(\frac{C_4}{C} - \frac{B_4}{B}\right) = 0 \tag{2.12}$$

Since $B \neq C$, equation (2.12) gives

$$A = N \text{ (constant)} \tag{2.13}$$

From (2.10) and (2.13), we have

$$\frac{B_{44}}{B} + \frac{B_4C_4}{BC} = 0 \tag{2.14}$$

Again from equations (2.11) and (2.13), we get

$$\frac{B_{44}}{B} = \frac{C_{44}}{C}$$

which on integration gives

$$B_4C - BC_4 = k_2 \tag{2.15}$$

k_2 being a constant of integration. On substituting $B/C = \alpha, BC = \beta$ so that $B^2 = \alpha\beta$ and $C^2 = \beta/\alpha$

Equation (2.15) reduces to

$$\left(\frac{\alpha_4}{\alpha}\right)\beta = k_2 \tag{2.16}$$

From (2.14) we have

$$\left[\left(\frac{\alpha_4}{\alpha}+\frac{\beta_4}{\beta}\right)\beta\right]_4=0 \tag{2.17}$$

From (2.16) and (2.17) we get

$$\beta_{44}=0$$

which gives

$$\beta=k_3t+b \tag{2.18}$$

where k_3 and b are constants of integration.

From (2.16) and (2.18) we get

$$\alpha=k_4(k_3t+b)^{k_2/k_3}$$

Therefore,

$$B^2=k_4(k_3t+b)^{\left(1+\frac{k_2}{k_3}\right)} \tag{2.19}$$

and

$$C^2=\frac{1}{k_4}(k_3t+b)^{\left(1-\frac{k_2}{k_3}\right)} \tag{2.20}$$

Consequently the line-element (2.1) takes the form

$$ds^2=-dt^2+N^2dx^2+k_4(k_3t+b)^{\left(1+\frac{k_2}{k_3}\right)}dy^2+\frac{1}{k_4}(k_3t+b)^{\left(1-\frac{k_2}{k_3}\right)}dz^2 \tag{2.21}$$

By the following transformation of coordinates

$$Nx\to x,k_4^{\frac{1}{2}}y\to y,k_4^{-\frac{1}{2}}z\to z$$

This line-element reduces to the form

$$ds^2=-dt^2+dx^2+(k_3t+b)^{\left(1+\frac{k_2}{k_3}\right)}dy^2+(k_3t+b)^{\left(1-\frac{k_2}{k_3}\right)}dz^2 \tag{2.22}$$

The pressure p and density ρ for the model (2.22) are given by

$$8\pi p=8\pi\rho=\frac{(k_3^2-k_2^2)}{4(k_3t+b)^2}\times\left[C_1\exp\left\{\sqrt{\frac{\lambda}{6}}\left(1-\frac{k_2^2}{k_3^2}\right)^{\frac{1}{2}}\log(k_3t+b)\right\}+\right.$$

$$\left.C_2\exp\left\{-\sqrt{\frac{\lambda}{6}}\left(1-\frac{k_2^2}{k_3^2}\right)^{\frac{1}{2}}\log(k_3t+b)\right\}\right] \tag{2.23}$$

where C_1 and C_2 are constants of integration.

Also the scalar field ϕ is given by

$$\phi = \left[C_1 \exp\left\{\sqrt{\frac{\lambda}{6}}\left(1 - \frac{k_2^2}{k_3^2}\right)^{\frac{1}{2}} \log(k_3 t + b)\right\}\right.$$

$$\left. + C_2 \exp\left\{-\sqrt{\frac{\lambda}{6}}\left(1 - \frac{k_2^2}{k_3^2}\right)^{\frac{1}{2}} \log(k_3 t + b)\right\}\right] \tag{2.24}$$

For the reality of p and ρ and the condition

$$p > 0, \rho > 0 \text{ to hold when } k_2^2 < k_3^2.$$

The volume element of the model (2.22) is given by

$$V = (-g)^{\frac{1}{2}} = (k_3 t + b) \tag{2.25}$$

Thus, the volume increases as the time increases i.e., the model is expanding with time. The non-vanishing components of the Weyl's conformal curvature tensor C_{hi}^{jk} are

$$C_{14}^{14} = C_{23}^{23} = \frac{(k_2^2 - k_3^2)}{6(k_3 t + b)^2}$$

$$C_{12}^{12} = C_{34}^{34} = \frac{(k_3^2 - k_2^2)}{12(k_3 t + b)^2} \tag{2.26}$$

$$C_{13}^{13} = C_{24}^{24} = \frac{(k_3^2 - k_2^2)}{12(k_3 t + b)^2}$$

The flow vector V^i is given by

$$V^1 = V^2 = V^3 = 0, V^4 = 1 \tag{2.27}$$

It satisfies $\bar{v}_j^i; \bar{v}^j = 0$, so that the flow is geodetic.
Also $W_i j = 0$.

The scalar of expansion θ is

$$\theta = \frac{k_3}{3(k_3 t + b)} \tag{2.28}$$

The non-zero components of shear tensor σ_{ij} are :

$$\sigma_{11} = \frac{k_3}{3(k_3 t + b)}$$

$$\sigma_{22} = \frac{(3k_2 + k_3)}{6}(k_3 t + b)^{\frac{k_2}{k_3}} \tag{2.29}$$

$$\sigma_{33} = \frac{(3k_2 - k_3)}{6(k_3 t + b)^{\frac{k_2}{k_3}}}$$

$$\sigma_{44} = \frac{2k_3}{3(k_3t+b)}$$

and the shear σ is

$$\sigma^2 = \frac{(9k_2^2 + 11k_3^2)}{36(k_3t+b)^2} \tag{2.29}$$

Thus, the model represents an irrotational, expanding universe with shear.

3 Discussion

When $\lambda \to 0$, the scalar field (ϕ) from equation (2.24) becomes constant and hence model (2.22) represents general relativistic anisotropic Bianchi type-I universe discussed by Roy and Prakash [6]. Model obtained in this paper is new and like other models with $p = \rho$ they may be used in relativistic cosmology for the description of very early stages of the universe expansion.

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On θ-CENTRALIZERS OF SEMIPRIME RINGS

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Abstract

The main result of the present article is the following: Let R be a 2-torsion-free semiprime ring, θ be an endomorphism of R and T: $R \to R$ be an additive mapping such that $T(xyx) = \theta(x)T(y)\theta(x)$ holds for all $x, y \in R$. Then T is a $\theta-$centralizer of R.

1 Introduction

This note has been motivated by the works of J. Vukman [4] and E. Albaş [1]. Throughout, R will represent an associative ring with center $Z(R)$, not necessatily with an identity element. A ring R is 2-torsion-free, if $2x = 0$, $x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ for $x, y \in R$ will be denoted by $[x, y]$. We shall use basic commutator

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identities $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = [x, z]y + x[y, z]$, for $x, y \in R$. Recall that R is semiprime if $aRa = (0)$ implies $a = 0$, for every $a \in R$.

B. Zalar [5] introduced the following notion. Let R be a semiprime ring. A *left* (resp. *right*) *centralizer* of R is an additive mapping $T\colon R \to R$ satisfying $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in R$. If T is a left and a right centralizer then T is a *centralizer*. In case R has an identity element, $T\colon R \to R$ is a left (resp. right) centralizer if and only if T is of the form $T(x) = ax$ (resp. $T(x) = xa$) for some fixed element $a \in R$. An additive mapping $T\colon R \to R$ is called a *left* (resp. *right*) *Jordan centralizer* in case $T(x^2) = T(x)x$ (resp. $T(x^2) = xT(x)$) holds for $x \in R$, and is called a *Jordan centralizer* if T satisfies $T(xy+yx) = T(x)y+yT(x) = T(y)x+xT(y)$ for all $x, y \in R$. In [5], it was shown that a Jordan centralizer of a semiprime ring is a left centralizer, and each Jordan centralizer is a centralizer.

Following ideas from M. Brešar [2], B. Zalar [5] has proved that any left (right) Jordan centralizer on a 2-torsion-free semiprime ring is a left (right) centralizer.

If $T\colon R \to R$ is a centralizer, where R is an arbitrary ring, then T satisfies the relation

$$T(xyx) = xT(y)x, \ \forall\, x, y \in R. \tag{1}$$

It seems natural to ask whether the converse is true. More precisely, asking for whether an additive mapping T on a ring R satisfying relation (1) is a centralizer. In [4], J. Vukman proved that the answer is affirmative in case R is a 2-torsion-free semiprime ring. The proof of his result is rather long, but it is elementary in the sense that it requires no specific knowledge concerning semiprime ring theory in order to follow the proof.

Recently, E. Albaş [1] introduced the following definitions, which are generalizations of the definitions of centralizer and Jordan centralizer. Let R be a semiprime 2-torsion-free ring, and let θ be an endomorphism of R. A *Jordan θ-centralizer* of R is an additive mapping $f\colon R \to R$ satisfying $f(xy+yx) = f(x)\theta(y)+\theta(y)f(x) = f(y)\theta(x)+\theta(x)f(y)$ for all $x, y \in R$. An additive mapping $f\colon R \to R$ is called a *left* (resp. *right*) *θ-centralizer* of R if $f(xy) = f(x)\theta(y)$ (resp. $f(xy) = \theta(x)f(y)$) for all $x, y \in R$. If f is a left and right θ-centralizer then it is natural to call f a *θ-centralizer*. It is clear that for an additive mapping $T\colon R \to R$ associated with a homomorphism $\theta\colon R \to R$, if $L_a(x) = a\theta(x)$ and $R_a(x) = \theta(x)a$ for a fixed element $a \in R$ and for all $x \in R$, then L_a is a left $\theta-$centralizer andR_a is a right $\theta-$centralizer. Clearly every centralizer is a special case of a θ-centralizer with $\theta = id_R$.

An additive mapping $f\colon R \to R$ is called a *left* (resp. *right*) *Jordan θ-centralizer* of R if $f(x^2) = f(x)\theta(x)$ (resp. $f(x^2) = \theta(x)f(x)$) for all $x \in R$. It is clear that a left θ-centralizer of R is a left Jordan θ-centralizer and, analogously, a θ-centralizer of R is a Jordan θ-centralizer of R. The converse is no longer true, in general. In [1], E. Albaş proved, under some conditions, that in a 2-torsion-free semiprime ring R, every Jordan θ-centralizer is a θ-centralizer. In [3], W. Cortes and C. Haetinger proved this question changing the semiprimality condition on R. The main result of this paper is the following: Let R be a 2-torsion-free ring which has a commutator right (resp. left) nonzero divisor and let $G\colon R \to R$ be a left (resp. right) Jordan σ-centralizer mapping of R, where σ is an automorphism of R. Then G is a left (resp. right) σ-centralizer mapping of R.

Now, if $T\colon R \to R$ is a θ-centralizer associated with a function $\theta\colon R \to R$, where R is an arbitrary ring, then T satisfies the relation

$$T(xyx) = \theta(x)T(y)\theta(x) \quad \forall\, x, y \in R. \tag{2}$$

Again, as J. Vukman [4] did on the centralizer case, we are asking whether an additive mapping T on a ring R satisfying relation (2) is a θ-centralizer for every $x, y \in R$. It is the aim in this paper to prove that the answer is affirmative in case R is a 2-torsion-free semiprime ring with some conditions on θ.

Otherwise unless stated, R will be a 2-torsion-free semiprime rings, and θ an endomorphism of R.

2 Results

The main goal of this paper is to prove the following

Theorem 2.1 *Let R be a 2-torsion-free semiprime ring and let $T\colon R \to R$ be an additive mapping such that $T(xyx) = \theta(x)T(y)\theta(x)$ holds for all pairs $x, y \in R$, where θ is a nonzero surjective endomorphism on R with $\theta(Z(R)) = Z(R)$. Then T is a θ-centralizer.*

Note that if we put $y = x$ in relation (2) it gives

$$T(x^3) = \theta(x)T(x)\theta(x), \quad \forall\, x \in R. \tag{3}$$

The question arises whether in a 2-torsion-free semiprime ring the above relation implies that T is a $\theta-$centralizer.

We shall prove that the answer is affirmative in case R has an identity element.

Theorem 2.2 *Let R be a 2-torsion-free semiprime ring with an identity element, θ a nonzero surjective homomorphism on R, and let $T\colon R \to R$ be an additive mapping such that $T(x^3) = \theta(x)T(x)\theta(x)$ holds for all $x \in R$. Then T is a θ-centralizer.*

3 Proofs

For the proof of Theorem 2.1 the following lemma will be needed.

Lemma 3.1 [4, Lemma 1] *Let R be a semiprime ring. Suppose that the relation $axb + bxc = 0$ holds for all $x \in R$ and some $a, b, c \in R$. In this case $(a + c)xb = 0$ is satisfied for all $x \in R$.*

Proof of Theorem 2.1. To prove that T is a θ-centralizer of R, we intend to prove the relation

$$[T(x), \theta(x)] = 0, \; \forall x \in R. \tag{4}$$

For the proof of the above relation we shall need the weaker relation below

$$[[T(x), \theta(x)], \theta(x)] = 0, \; \forall x \in R. \tag{5}$$

Replacing x by $x+z$ in (2), we get

$$T(xyz+zyx)=\theta(x)T(y)\theta(z)+\theta(z)T(y)\theta(x), \quad \forall x,y,z\in R. \tag{6}$$

Putting $y=x$ and $z=y$ in (6) one obtain

$$T(x^2y+yx^2)=\theta(x)T(x)\theta(y)+\theta(y)T(x)\theta(x), \quad \forall\, x,y\in R. \tag{7}$$

For $z=x^3$, relation (6) reduces to

$$T(xyx^3+x^3yx)=\theta(x)T(y)\theta(x^3)+\theta(x^3)T(y)\theta(x), \quad \forall\, x,y\in R. \tag{8}$$

Now replace y by xyx in (7). We get

$$T(xyx^3+x^3yx)=\theta(x)T(x)\theta(xyx)+\theta(xyx)T(x)\theta(x), \quad \forall\, x,y\in R. \tag{9}$$

The substitution x^2y+yx^2 for y in relation (2) gives

$$T(xyx^3+x^3yx)=\theta(x)T(x^2y+yx^2)\theta(x),\ \forall\, x,y\in R.$$

Which implies, because of (7),

$$T(x^3yx+xyx^3)=\theta(x^2)T(x)\theta(yx)+\theta(xy)T(x)\theta(x^2), \quad \forall\, x,y\in R. \tag{10}$$

Combining (9) with (10) we arrive at

$$\theta(x)[T(x),\theta(x)]\theta(yx)-\theta(xy)[T(x),\theta(x)]\theta(x)=0, \quad \forall\, x,y\in R. \tag{11}$$

Putting in equation (11), $a=\theta(x)[T(x),\theta(x)]$, $b=\theta(x)$, $c=-[T(x),\theta(x)]\theta(x)$ and $z=\theta(y)$, this expression can be rewritten on the form $azb+bzc=0$, for every $z\in R$. Applying Lemma 3.1 on the above relation it follows that

$$[[T(x),\theta(x)],\theta(x)]\theta(yx)=0, \quad \forall\, x,y\in R. \tag{12}$$

Let $\theta(y)$ be $\theta(y)[T(x),\theta(x)]$ in (12). We have

$$[[T(x),\theta(x)],\theta(x)]\theta(y)[T(x),\theta(x)]\theta(x)=0, \quad \forall\, x,y\in R. \tag{13}$$

Right multiplication of (12) by $[T(x),\theta(x)]$ gives

$$[[T(x),\theta(x)],\theta(x)]\theta(y)\theta(x)[T(x),\theta(x)]=0, \quad \forall\, x,y\in R. \tag{14}$$

Subtracting (14) from (13) we obtain

$$[[T(x),\theta(x)],\theta(x)]\theta(y)[[T(x),\theta(x)],\theta(x)]=0, \quad \forall\, x,y\in R. \tag{15}$$

Since R is semiprime and θ is onto we get, $[[T(x),\theta(x)],\theta(x)]=0$, for all $x\in R$.

The next step is to prove the relation

$$\theta(x)[T(x),\theta(x)]\theta(x)=0, \quad \forall\, x\in R. \tag{16}$$

Substituting x by $x+y$ in (5) we have, for every $x, y \in R$, $[[T(x), \theta(x)], \theta(y)] + [[T(x), \theta(y)], \theta(x)] + [[T(y), \theta(y)], \theta(x)] + [[T(y), \theta(x)], \theta(y)] + [[T(y), \theta(x)], \theta(x)] + [[T(x), \theta(y)], \theta(y)] = 0$. Putting $-x$ for x in the above relation and comparing the expression so obtained with the above one we get for every $x, y \in R$

$$[[T(x), \theta(x)], \theta(y)] + [[T(x), \theta(y)], \theta(x)] + [[T(y), \theta(x)], \theta(x)] = 0. \tag{17}$$

Replacing y by xyx in (17) and using (2), (5) and (17) we obtain

$$\begin{aligned}
0 &= [[T(x), \theta(x)], \theta(xyx)] + [[T(x), \theta(xyx)], \theta(x)] + \\
&\quad + [[\theta(x)T(y)\theta(x), \theta(x)], \theta(x)] = \\
&= \theta(x)[[T(x), \theta(x)], \theta(y)]\theta(x) + \\
&\quad + [[T(x), \theta(x)]\theta(yx) + \theta(x)[T(x), \theta(y)]\theta(x) + \theta(xy)[T(x), \\
&\quad \theta(x)], \theta(x)] + [\theta(x)[T(y), \theta(x)]\theta(x), \theta(x)] = \\
&= \theta(x)[[T(x), \theta(x)], \theta(y)]\theta(x) + [T(x), \theta(x)][\theta(y), \theta(x)]\theta(x) + \\
&\quad + \theta(x)[[T(x), \theta(y)], \theta(x)]\theta(x) + \theta(x)[\theta(y), \theta(x)][T(x), \theta(x)] + \\
&\quad + \theta(x)[[T(y), \theta(x)], \theta(x)]\theta(x) = \\
&= [T(x), \theta(x)][\theta(y), \theta(x)]\theta(x) + \theta(x)[\theta(y), \theta(x)][T(x), \theta(x)] = \\
&= [T(x), \theta(x)]\theta(yx^2) - \theta(x^2y)[T(x), \theta(x)] + \\
&\quad + \theta(xyx)[T(x), \theta(x)] - [T(x), \theta(x)]\theta(xyx).
\end{aligned}$$

Therefore, for every $x, y \in R$, we have

$$[T(x), \theta(x)]\theta(yx^2) - \theta(x^2y)[T(x), \theta(x)] + \theta(xyx)[T(x), \theta(x)] - [T(x), \theta(x)]\theta(xyx) = 0.$$

Which reduces because of (5) and (11) to

$$[T(x), \theta(x)]\theta(yx^2) - \theta(x^2y)[T(x), \theta(x)] = 0, \ \forall\, x, y \in R.$$

Left multiplication of the above relation by $\theta(x)$ gives

$$\theta(x)[T(x), \theta(x)]\theta(yx^2) - \theta(x^3y)[T(x), \theta(x)] = 0, \ \forall\, x, y \in R.$$

One can replace in the above relation, according to (11), $\theta(x)[T(x), \theta(x)]\theta(yx)$ by $\theta(xy)[T(x), \theta(x)]\theta(x)$, which gives

$$\theta(xy)[T(x), \theta(x)]\theta(x^2) - \theta(x^3y)[T(x), \theta(x)] = 0, \quad \forall\, x, y \in R. \tag{18}$$

Left multiplication of the above relation by $T(x)$ gives

$$T(x)\theta(xy)[T(x), \theta(x)]\theta(x^2) - T(x)\theta(x^3y)[T(x), \theta(x)] = 0, \quad \forall\, x, y \in R. \tag{19}$$

Substitute $T(x)\theta(y)$ for $\theta(y)$ in (18) which leads to

$$\theta(x)T(x)\theta(y)[T(x), \theta(x)]\theta(x^2) - \theta(x^3)T(x)\theta(y)[T(x), \theta(x)] = 0, \quad \forall\, x, y \in R. \tag{20}$$

Subtracting (20) from (19) we obtain for all $x, y \in R$

$$[T(x), \theta(x)]\theta(y)[T(x), \theta(x)]\theta(x^2) - [T(x), \theta(x^3)]\theta(y)[T(x), \theta(x)] = 0. \tag{21}$$

Which can be rewritten in the form
$[T(x),\theta(x^3)]\theta(y)[T(x),\theta(x)] - [T(x),\theta(x)]\theta(y)[T(x),\theta(x)]\theta(x^2) = 0, \forall\, x,y \in R.$

If we take $a = [T(x),\theta(x^3)]$, $b = [T(x),\theta(x)]$, $c = -[T(x),$
$\theta(x)]\theta(x^2)$ and $z = \theta(y)$ in the above relation, it can be rewritten in the form $azb+bzc = 0$, for every $z \in R$. Applying Lemma 3.1 again, it follows that

$$([T(x),\theta(x^3)] - [T(x),\theta(x)]\theta(x^2))\theta(y)[T(x),\theta(x)] = 0, \quad \forall\, x,y \in R. \tag{22}$$

Which reduces for every $x, y \in R$ to

$$(\theta(x)[T(x),\theta(x)]\theta(x) + \theta(x^2)[T(x),\theta(x)])\theta(y)[T(x),\theta(x)] = 0. \tag{23}$$

Relation (5) makes it possible now to write $[T(x),\theta(x)]\theta(x)$ instead of $\theta(x)[T(x),\theta(x)]$, which means that, in the above expression, $\theta(x^2)[T(x), \theta(x)]$ can be replaced by $\theta(x)[T(x),\theta(x)]\theta(x)$. Thus we have, for every $x, y \in R$,

$$\theta(x)[T(x),\theta(x)]\theta(xy)[T(x),\theta(x)] = 0.$$

Right multiplication of the above relation by $\theta(x)$ and substituting $\theta(yx)$ for $\theta(y)$ gives finally $\theta(x)[T(x),\theta(x)]\theta(xyx)[T(x),\theta(x)]\theta(x) = 0$, for every x, y belonging to R. By the the semiprimeness of R and the surjectivity of θ we have that $\theta(x)[T(x),\theta(x)]\theta(x) = 0$ holds for every $x \in R$, and so (16) follows.

Next we prove the following relation

$$\theta(x)[T(x),\theta(x)] = 0, \quad \forall\, x \in R. \tag{24}$$

The substitution of yx for y in (11) gives, because of (16),

$$\theta(x)[T(x),\theta(x)]\theta(yx^2) = 0, \quad \forall\, x,y \in R. \tag{25}$$

Putting $\theta(y)T(x)$ for $\theta(y)$ in the above relation we obtain

$$\theta(x)[T(x),\theta(x)]\theta(y)T(x)\theta(x^2) = 0, \quad \forall\, x,y \in R. \tag{26}$$

Right multiplication of (25) by $T(x)$ gives

$$\theta(x)[T(x),\theta(x)]\theta(yx^2)T(x) = 0, \quad \forall\, x,y \in R. \tag{27}$$

Subtracting (27) from (26) we obtain $\theta(x)[T(x),\theta(x)]\theta(y)[T(x),$
$\theta(x^2)] = 0$, for every $x, y \in R$, which can be rewritten in the form
$\theta(x)[T(x),\theta(x)]\theta(y)([T(x),\theta(x)]\theta(x) + \theta(x)[T(x),\theta(x)]) = 0,$
$\forall\, x, y \in R.$

According to (5) we can replace $[T(x),\theta(x)]\theta(x)$ in the relation above by $\theta(x)[T(x),\theta(x)]$, which gives $\theta(x)[T(x),\theta(x)]\theta(yx)[T(x),\theta(x)] = 0$, for all $x, y \in R$. So, by the surjectivity of θ and the semiprimeness of R we get $\theta(x)[T(x),\theta(x)] = 0$, for each $x \in R$. Whence relation (24) holds. It follows from (5) and (24) that

$$[T(x),\theta(x)]\theta(x) = 0, \ \forall\, x \in R.$$

Substituting x by $x+y$ in the expression above, we obtain for all $x, y \in R$ that

$$[T(x), \theta(x)]\theta(y) + [T(x), \theta(y)]\theta(x) + [T(x), \theta(y)]\theta(y)+$$

$$[T(y), \theta(x)]\theta(x) + [T(y), \theta(x)]\theta(y) + [T(y), \theta(y)]\theta(x) = 0.$$

Replacing now x by $-x$ in this equation and comparing the relation so obtained with the above one we arrive at. $[T(x), \theta(x)]\theta(y) + [T(x), \theta(y)]\theta(x) + [T(y), \theta(x)])\theta(x) = 0$, for every $x, y \in R$.

Right multiplication of the last expression by $[T(x), \theta(x)]$ gives, because of (24), $[T(x), \theta(x)]\theta(y)[T(x), \theta(x)] = 0$, for all $x, y \in R$. So, by the surjectivity of θ and the semiprimeness of R we get (4).

Let now $A(x, y)$ stands for $T(xy+yx) - T(y)\theta(x) - \theta(x)T(y)$. Our next task is to prove the following relation

$$T(xy+yx) = T(y)\theta(x) + \theta(x)T(y), \quad \forall\, x \in R. \tag{28}$$

In order to prove it we need the relations below

$$\theta(x)A(x, y)\theta(x) = 0, \quad \forall\, x \in R, \tag{29}$$

and

$$[A(x, y), \theta(x)] = 0, \quad \forall\, x \in R. \tag{30}$$

Let us first prove relation (29). The substitution $xy+yx$ for y in (2) gives

$$T(x^2yx + xyx^2) = \theta(x)T(xy+yx)\theta(x), \quad \forall\, x, y \in R. \tag{31}$$

On the other hand we obtain, by putting $z = x^2$ in (6),

$$T(x^2yx + xyx^2) = \theta(x)T(y)\theta(x^2) + \theta(x^2)T(y)\theta(x), \quad \forall\, x, y \in R. \tag{32}$$

By comparing (31) and (32) we arrive at (29).

Substituting x by $x+z$ in relation (29) and using (29) again we get for every $x, y, z \in R$ that

$$\theta(x)A(x, y)\theta(z) + \theta(x)A(z, y)\theta(x) + \theta(x)A(z, y)\theta(z)$$

$$+\theta(z)A(x, y)\theta(x) + \theta(z)A(x, y)\theta(z) + \theta(z)A(z, y)\theta(x) = 0$$

Putting now $-x$ for x in this expression and comparing the relation so obtained with the above one, we obtain $\theta(x)A(x, y)\theta(z) + \theta(x)A(z, y)\theta(x) + \theta(z)A(x, y)\theta(x) = 0$, for every $x, y, z \in R$. Right multiplication of this relation by $A(x, y)\theta(x)$ gives, because of (29),

$$\theta(x)A(x, y)\theta(z)A(x, y)\theta(x) = 0, \quad \forall\, x, y, z \in R. \tag{33}$$

Now, let us proving relation (30). The linearization of (4) gives

$$[T(x), \theta(y)] + [T(y), \theta(x)] = 0, \quad \forall\, x, y \in R. \tag{34}$$

Putting $xy+yx$ for y in the above relation and using (4) we obtain
$[T(x), \theta(xy+yx)] + [T(xy+yx), \theta(x)] = \theta(x)[T(x), \theta(y)] + [T(x), \theta(y)]\theta(x)+$

$[T(xy+yx),\theta(x)]=0$, for all $x,y\in R$. Thus we have $[T(xy+yx),\theta(x)]+\theta(x)[T(x),\theta(y)]+[T(x),\theta(y)]\theta(x)=0$, for all $x,y\in R$. According to (34) we can replace $[T(x),\theta(y)]$ by $-[T(y),\theta(x)]$ in this expression. Therefore, $[T(xy+yx),\theta(x)]-\theta(x)[T(y),\theta(x)]-[T(y),\theta(x)]\theta(x)=0$, for all $x,y\in R$, which can be rewritten in the form $[T(xy+yx)-T(y)\theta(x)-\theta(x)T(y),\theta(x)]=0$, for every $x,y\in R$. The proof of relation (30) is therefrom complete.

Relation (30) makes it possible to replace in (33) $\theta(x)A(x,y)$ by $A(x,y)\theta(x)$. Thus we have

$$A(x,y)\theta(x)\theta(z)A(x,y)\theta(x)=0, \quad \forall\, x,y,z\in R, \tag{35}$$

whence, by the surjectivity of θ and the semiprimeness of R, it follows that

$$A(x,y)\theta(x)=0, \quad \forall\, x,y\in R. \tag{36}$$

Of course we also have,

$$\theta(x)A(x,y)=0, \quad \forall\, x,y\in R. \tag{37}$$

The linearization of (36) with respect to x gives $A(x,y)\theta(z)+A(z,y)\theta(x)=0$, for all $x,y,z\in R$.

Right multiplication of the above relation by $A(x,y)$ gives, because of (37), $A(x,y)\theta(z)A(x,y)=0$, for all $x,y,z\in R$, which, by the surjectivity of θ and the semiprimeness of R, gives $A(x,y)=0$, for every $x,y\in R$. The proof of relation (28) is therefrom complete, too.

In particular for $x=y$ relation (30) reduces to $2T(x^2)=T(x)\theta(x)+\theta(x)T(x)$, for all $x\in R$.

Combining the above relation with (4) we arrive at $T(x^2)=T(x)\theta(x)$, for all $x\in R$, and $T(x^2)=\theta(x)T(x)$, for every $x\in R$, since R is 2-torsion-free.

By [1, Theorem 2] it follows that T is a left and also right $\theta-$centralizer, which completes the proof.

In particular, we get [4, Theorem 1] as a corollary.

Corollary 3.2 *Let R be a 2-torsion free semiprime ring and let $T\colon R\to R$ be an additive mapping. Suppose that $T(xyx)=xT(y)x$ holds for all $x,y\in R$. In this case T is a centralizer.*

We conclude by proving Theorem 2.2.

Proof of Theorem 2.2. Let 1 denote the identity element of R. By assumption, relation (3) holds for every $x\in R$. Putting $x+1$ for x in (3) we obtain, for every $x\in R$,

$$3T(x^2)+2T(x)=T(x)\theta(x)+\theta(x)T(x)+\theta(x)a\theta(x)+a\theta(x)+\theta(x)a, \tag{38}$$

where a stands for $T(1)$. Replacing x by $-x$ in (38) and comparing the relation so obtained with the above one, we obtain

$$6T(x^2)=2T(x)\theta(x)+2\theta(x)T(x)+2\theta(x)a\theta(x), \ \forall\, x\in R. \tag{39}$$

From (39) and since R is 2-torsion-free we have

$$3T(x^2) = T(x)\theta(x) + \theta(x)T(x) + \theta(x)a\theta(x), \ \forall\, x \in R.$$

Substituting from the above relation in (38) we get

$$2T(x) = a\theta(x) + \theta(x)a, \ \forall\, x \in R. \tag{40}$$

We intend to prove that $a \in Z(R)$. According to (40) one can replace $2T(x)$ on the RHS of (39) by $a\theta(x) + \theta(x)a$ and $6T(x^2)$ on the LHS by $3a\theta(x^2) + 3\theta(x^2)a$, to get

$$a\theta(x^2) + \theta(x^2)a - 2\theta(x)a\theta(x) = 0, \ \forall\, x \in R.$$

The above relation can be rewritten in the form

$$[[a, \theta(x)], \theta(x)] = 0, \ \forall\, x \in R. \tag{41}$$

The linearaization of (41) gives

$$[[a, \theta(x)], \theta(y)] + [[a, \theta(y)], \theta(x)] = 0, \ \forall\, x, y \in R. \tag{42}$$

Putting xy for y in (42) we obtain, because of (41) and (42) that, for every $x, y \in R$,

$$\begin{aligned} 0 &= [[a, \theta(x)], \theta(xy)] + [[a, \theta(xy)], \theta(x)] = \\ &= [[a, \theta(x)], \theta(x)]\theta(y) + \theta(x)[[a, \theta(x)], \theta(y)] + \\ &\quad + [[a, \theta(x)]\theta(y), \theta(x)] + [\theta(x)[a, \theta(y)], \theta(x)] = \\ &= \theta(x)[[a, \theta(x)], \theta(y)] + [[a, \theta(x)], \theta(x)]\theta(y) + \\ &\quad + [a, \theta(x)][\theta(y), \theta(x)] + \theta(x)[[a, \theta(y)], \theta(x)] = \\ &= [a, \theta(x)][\theta(y), \theta(x)]. \end{aligned}$$

Thus we have $[a, \theta(x)][\theta(y), \theta(x)] = 0$, for each $x, y \in R$. The substitution $\theta(y)a$ for $\theta(y)$ on this relation gives $[a, \theta(x)]\theta(y)[a, \theta(x)] = 0$, for all $x, y \in R$. So, by the semiprimeness of R and the surjectivity of θ it follows $a \in Z(R)$, which reduces (40) to the form $T(x) = a\theta(x)$, for every $x \in R$. The proof is now complete.

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A CLASS OF α- UNIFORMLY ANALYTIC FUNCTIONS

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Abstract

Let $TS_s^*R_\lambda(\alpha, \mu, z_0)$ denote the class of univalent analytic functions having negative coefficients with two fixed points which are α- starlike functions involving Ruscheweyh derivatives. We determine the coefficient inequality, distortion theorem, extreme points and radius of starlikeness for the class $TS_s^*R_\lambda(\alpha, \mu, z_0)$. Also, the analogous results are obtained for the class $TC_sR_\lambda(\alpha, \mu, z_0)$, the class of α- uniformly convex functions involving Ruscheweyh derivatives.

1 Introduction

Let S denote the class of functions of the form $f(z) = a_1 + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $U : |z| < 1$. Let T denote the subclass of S consisting of functions whose non-zero coefficients from second on, are negative; that is, an analytic and univalent functions f is in T if and only if it can be expressed as

$$f(z) = a_1 z - \sum_{n=2}^{\infty} |a_n| z^n \tag{1.1}$$

for which either $f(z_0) = z_0$ or $f'(z_0) = 1$ with $-1 < z_0 < 1$.

Let the subclass T_λ consist of the functions f in T satisfying

$$(1-\lambda)\frac{f(z_0)}{z_0} + \lambda f'(z_0) = 1 \qquad (0 \le \lambda \le 1; z_0 \neq 0). \tag{1.2}$$

Keywords and phrases : Analytic functions, univalent functions, starlike functions, α- uniformly starlike functions, α- uniformly convex functions, Ruscheweyh derrivates, radius of starlikeness.

AMS Subject Classification : Primary 30C45, 30C50, 30C55.

For $0 \leq \alpha \leq 1$ and integer $\mu > -1$, consider the class $TS_s^* R_\lambda(\alpha, \mu, z_0)$ of univalent analytic functions which are α- uniformly starlike involving Ruscheweyh derivates with respect to symmetric points and consists of function f in T_λ satisfying the condition

$$Re\left[\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)}\right] > \alpha\left|\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)} - 1\right|, \tag{1.3}$$

where the operator $D^\mu f$ is the Ruscheweyh derivate of f defined by

$$\begin{aligned} D^\mu f(z) &= \frac{z(z^{\mu-1})f(z))^\mu}{\mu!} = \frac{z}{(1-z)^{\mu+1}} * f(z) \\ &= a_1 z - \sum_{k=1}^{\infty} A_n(\mu)|a_n|z^n \end{aligned}$$

with

$$A_n(\mu) = \left(\begin{array}{c} n+\mu-1 \\ \mu \end{array}\right) = \frac{(\mu+1)(\mu+2)............(\mu+n-1)}{(n-1)!}.$$

Further, $f \in T_\lambda$ is in the class $TC_s R_\lambda(\alpha, \mu, z_0)$, the class of α- uniformly convex functions with symmetric points, if and only if $zf' \in TS_s^* R_\lambda(\alpha, \mu, z_0)$.

Rusheweyh derivates were introduced in [6]. One may refer [1] for uniformly starlike, to [4] [5] for α- uniformly starlike and to [8] for analytic functions with negative coefficients, also see [2], [3] together with references there in.

In this paper, we discuss coefficient inequality, distortion theorem, extreme points and radius of starlikeness for the $TS_s^* R_\lambda(\alpha, \mu, z_0)$ and the analogous results for the class $TC_s R_\lambda(\alpha, \mu, z_0)$.

2 Coefficient Inequality

Theorem 2.1. Let $f \in T_\lambda$ is in the class $TS_s^* R_\lambda(\alpha, \mu, z_0)$ if and only if

$$\sum_{n=2}^{\infty}\{n(1+\alpha) - \alpha(1-(-1^n))A_n(\mu) - (1-\alpha)(1-\lambda+n\lambda)z_0^{n-1}\}|a_n| < 1-\alpha. \tag{2.1}$$

Proof. Let $f \in TS_s^* R_\lambda(\alpha, \mu, z_0)$. Using the fact that $\mathrm{Re}(\omega) > \alpha|\omega - 1|$ if and only if $Re[\omega(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma}] > 1$ for real λ, and letting $\omega = \frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)}$ in (1.3), we get

$$Re\left[\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)}(1+\alpha e^{i\lambda}) - \alpha e^{i\gamma}\right] > 0,$$

which on simplification gives

$$\left(za_1 - \sum_{n=2}^{\infty} n|a_n|A_n(\mu)z^n\right)(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma}\left(2a_1 z - \sum_{n=2}^{\infty}(1-(-1)^n)A_n(\mu)|a_n|z^n\right) > 0.$$

The above inequality holds for all z in U. Letting $z \to 1^-$, we have

$$a_1(1-\alpha) - \sum_{n=2}^{\infty}\{n(1+\alpha) - \alpha(1-(-1^n)\}A_n(\mu)|a_n| > 0. \tag{2.2}$$

Moreover, from equation (1.2), we obtain the value of a_1 as

$$a_1 = 1 + \sum_{n=2}^{\infty}(1-\lambda+n\lambda)|a_n|z_0^{n-1},$$

which on substituting in (2.2) gives

$$\sum_{n=2}^{\infty}\{(n(1+\alpha) - \alpha(1-(-1^n))A_n(\mu) - (1-\alpha)(1-\lambda+n\lambda)z_0^{n-1}\}|a_n| < 1-\alpha.$$

This verifies the inequality (2.1)

Conversely , suppose that the inequality (2.1) holds. We will now show that (1.3) is satisfied which in turn verifies that $f \in TS_s^*R_\lambda(\alpha,\mu,z_0)$. Using the fact that $Re(\omega) < \delta$ if and only if $|\omega - (1+\delta)| < |\omega + (1-\delta)|$, it is enough to show that

$$\left|\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)} - \left[1 + \alpha\left|\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)} - 1\right|\right]\right|$$

$$< \quad \left|\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)} + \left[1 - \alpha\left|\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)} - 1\right|\right]\right|$$

Put $e^{i\gamma} = \frac{D^\mu f(z)}{|D^\mu f(z) - D^\mu f(-z)|}$. Then note that

$$\begin{aligned} E &= \left|\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu f(-z)} + 1 - \alpha\left|\frac{z(D^\mu f(z))'}{D^\mu f(z)} - 1\right|\right| \\ &> \frac{|z|}{|D^\mu f(z) - D^\mu f(-z)|}\left[a_1(3-\alpha) - \sum_{n=2}^{\infty}\{n+1-(-1)^n\right. \\ &\quad \left. +\alpha(n-1+(-1)^n)\}A_n(\mu)|a_n|\right] \end{aligned}$$

and

$$\begin{aligned} F &= \left|\frac{z(D^\mu f(z))'}{D^\mu f(z) - D^\mu(-z)} - 1 - \alpha\left|\frac{z(D^\mu f(z))'}{D^\mu f(z)} - 1\right|\right| \\ &< \frac{|z|}{|D^\mu f(z) - D^\mu f(-z)|}\left[a_1(1+\alpha) + \sum_{n=2}^{\infty}\{n-1+(-1)^n\right. \\ &\quad \left. +\alpha(n-1+(-1)^n)\}A_n(\mu)|a_n|\right] \end{aligned}$$

so that by using (2.2) , we get $E-F\geq 0$.

Remark 2.2. The result in Theorem 2.1 is sharp. The extremal functions is given by

$$f(z)=\frac{\{n(1+\alpha)-\alpha(1-(-1^n)\}A_n(\mu)z-(1-\alpha)z^n}{\{n(1+\alpha)-\alpha(1-(-1)^n)\}A_n(\mu)-(1-\alpha)(1-\lambda+n\lambda)z_0^{n-1}},$$

$$n\geq 2.$$

3 Applications of Coefficients Theorem

In this section, we present distortion theorem, extreme points for the class $TS_s^*R_\lambda(\alpha,\mu,z_0)$ as consequences of the coefficient inequality established in Theorem 2.1.
Theorem 3.1. If $f\in TS_s^*R_\lambda(\alpha,\mu,z_0), then, for z\in U$

$$a_1\left[r-\frac{1-\alpha}{2(1+\alpha)(1+\mu)}r^2\right]$$

$$\leq\left|f(z)\right|\leq a_1\left[r+\frac{1-\alpha}{2(1+\alpha)(1+\mu)}r^2\right] \tag{3.1.1}$$

$$a_1\left[r-\frac{1-\alpha}{2(1+\alpha)}r^2\right]$$

$$\leq\left|D^\mu f(z)\right|\leq a_1\left[r+\frac{1-\alpha}{2(1+\alpha)}r^2\right] \tag{3.1.2}$$

and

$$a_1\left[1-\frac{1-\alpha}{(1+\alpha)(1+\mu)}r\right]$$

$$\leq|f'(z)|\leq a_1\left[1+\frac{1-\alpha}{(1+\alpha)(1+\mu)}r\right]. \tag{3.1.3}$$

Proof. Since $\{A_n(\mu)\}_{n\geq 2}$ is non-decreasing, in view of the inequality (2.2), we get

$$2(1+\alpha)(1+\mu)\sum_{n=2}^{\infty}|a_n|$$

$$\leq\{n(1+\alpha)-\alpha(1-(-1^n))\}A_n(\mu)|a_n|\leq a_1(1-\alpha),$$

which gives

$$\sum_{n=2}^{\infty}A_n(\mu)|a_n|\leq\frac{a_1(1-\alpha)}{2(1+\alpha)} \tag{3.1.4}$$

$$\sum_{n=2}^{\infty}|a_n|\leq\frac{a_1(1-\alpha)}{2(1+\alpha)(1+\mu)}. \tag{3.1.5}$$

Therefore, using (3.1.5) in (1.1), we get

$$|f(z)| \geq a_1\left[r - \frac{1-\alpha}{2(1+\alpha)(1+\mu)}r^2\right]$$

$$|f(z)| \leq a_1\left[r + \frac{1-\alpha}{2(1+\alpha)(1+\mu)}r^2\right].$$

This yields the result (3.1.1). Also, if $f \in TS_s^*R_\lambda(\alpha, \mu, z_0)$, using (3.1.4) in (1.3), we get the result (3.1.2).

Further, note that

$$|a_1| - r\sum_{n=2}^{\infty} n|a_n| \leq f'(z)| \leq |a_1| + r\sum_{n=2}^{\infty} n|a_n|.$$

But, in view of Theorem 2.1, we have

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{a_1(1-\alpha)}{(1+\alpha)(1+\mu)},$$

which in view of above, yields the result (3.1.3).
Theorem 3.2. Let $f_1(z) = a_1 z$ and

$$f_n(z) = a_1 z$$
$$-\frac{(1-\alpha)z^n}{\{n(1+\alpha) - \alpha(1-(-1)^n)\}A_n(\mu) - (1-\alpha)(1-\lambda+n\lambda)z_0^{n-1}},$$

where $k \geq 1$. Then $f \in TS_s^*R_\lambda(\alpha, \mu, z_0)$. if and only if it can be expressed in the form

$$f(z) = \sum_{n=2}^{\infty} d_n f_n(z), where\ d_n \geq 0\ and \sum_{n=2}^{\infty} d_n = 1.$$

Proof. Let us write

$$f(z) = \sum_{n=1}^{\infty} d_n f_n(z) = a_1 z$$
$$-\sum_{n=2}^{\infty} \frac{(1-\alpha)d_n z^n}{(n(1+\alpha) - \alpha(1-(-1)^n))A_n(\mu) - (1-\alpha)(1-\lambda+n\lambda)z_0^{n-1}}. \tag{3.2.1}$$

Put

$$d_n = \frac{\{(n(1+\alpha) - \alpha(1-(-1)^n))A_n(\mu) - (1-\alpha)(1-\lambda+n\lambda)\}|a_n|}{1-\alpha}.$$
$$n = 2, 3.....$$

and

$$d_1 = 1 - \sum_{n=2}^{\infty} d_n.$$

Now, first assume that $f \in TS_s^* R_\lambda(\alpha, \mu, z_0)$,. Then, by Theorem 2.1, we have

$$\sum_{n=2}^{\infty}\{(n(1+\alpha) - \alpha(1-(-1)^n))A_n(\mu)$$
$$-(1-\alpha)(1-\lambda+n\lambda)z_0^{n-1}\}|a_n| < 1-\alpha,$$

which gives

$$\sum_{n=2}^{\infty} d_n \leq 1 \quad and \quad d_1 \geq 0.$$

Conversely, let $\sum_{n=2}^{\infty} d_n \leq 1 \ \ and \ \ d_1 \geq 0$, which, by using in (3.2.1), verifies that

$$f(z) = a_1 z - \sum_{n=2}^{\infty} t_n z^n,$$

where

$$\sum_{n=2}^{\infty} t_n = \sum_{n=2}^{\infty}$$
$$\frac{(1-\alpha)d_n}{(n(1+\alpha) - \alpha(1-(-1)^n))A_n(\mu) - (1-\alpha)(1-\lambda+n\lambda)z_0^{n-1}}$$
$$\leq \sum_{n=2}^{\infty} d_n \leq 1.$$

Hence $f \in TS_s^* R_\lambda(\alpha, \mu, z_0)$.

Theorem 3.3. Let $f \in TS_s^* R_\lambda(\alpha, \mu, z_0)$. Then $f(z)$ is starlike in $|z| < r_\lambda(\alpha, \mu)$, where

$$r_\lambda(\alpha, \mu) = \inf_n \left\{ \frac{(n(1+\alpha) - \alpha(1-(-1)^n))A_n(\mu)}{n(1-\alpha)} \right\}^{\frac{1}{n-1}}$$

Proof. Noting that

$$\left| \frac{zf'(z)}{f(z)-f(-z)} - 1 \right| \leq \frac{a_1|z| + \sum_{n=2}^{\infty}(n-1+(-1)^n)|a_n||z|^n}{2a_1|z| - \sum_{n=2}^{\infty}(1-(-1)^n)|a_n||z|^n},$$

we find that $\left| \frac{zf'(z)}{f(z)-f(-z)} - 1 \right| < 1 \ \ for \ \ |z| < 1$
if $\sum_{n=2}^{\infty} n|a_n||z|^{n-1} < a_1$.

Hence f is starlike if

$$r_\lambda(\alpha, \mu) = \inf_n \left\{ \frac{n(1+\alpha) - \alpha(1-(-1)^n)}{n(1-\alpha)} A_n(\mu) \right\}^{\frac{1}{n-1}}, n = 2, 3.....$$

which completes the proof..

4 Class $TC_sR_\lambda(\alpha, \mu, z_0)$

In this section, using the fact that Let $f \in TC_sR_\lambda(\alpha, \mu, z_0)$, if and only if $zf' \in TS_s^*R_\lambda(\alpha, \mu, z_0)$, and the results proved for the class $TS_s^*R_\lambda(\alpha, \mu, z_0)$, in Sections 2 and 3, we obtain analogue results for the class $TC_sR_\lambda(\alpha, \mu, z_0)$, which are stated without proof.

Theorem 4.1. A function $f(z) = a_1z - \sum_{n=2}^{\infty}|a_n|z^n$ is in class $TC_sR_\lambda(\alpha, \mu, z_0)$, if and only if

$$\sum_{n=2}^{\infty} n\{(n(1+\alpha) - \alpha(1-(-1)^n))A_n(\mu)$$

$$-(1-\alpha)(1-\lambda+n\lambda)z_0^{n-1}\}|a_n| < 1-\alpha.$$

This result is sharp.

Theorem 4.2. If $f \in TC_sR_\lambda(\alpha, \mu, z_0)$, then, for $z \in U$,

$$a_1\left[r - \frac{1-\alpha}{4(1+\alpha)(1+\mu)}r^2\right]$$

$$\leq \left|f(z)\right| \leq a_1\left[r + \frac{1-\alpha}{4(1+\alpha)(1+\mu)}r^2\right] \tag{4.2.1}$$

$$a_1\left[r - \frac{1-\alpha}{4(1+\alpha)}r^2\right] \leq \left|D^\mu f(z)\right| \leq a_1\left[r + \frac{1-\alpha}{4(1+\alpha)}r^2\right] \tag{4.2.2}$$

and

$$a_1\left[1 - \frac{1-\alpha}{2(1+\alpha)(1+\mu)}r\right]$$

$$\leq \left|f'(z)\right| \leq a_1\left[1 + \frac{1-\alpha}{2(1+\alpha)(1+\mu)}r\right]. \tag{4.2.3}$$

Theorem 4.3. Let $f_1(z) = a_1z$ and

$$f_n(z) = a_1z$$

$$-\frac{(1-\alpha)z^n}{n[\{n(1+\alpha)-\alpha(1-(-1)^n)\}A_n(\mu) - (1-\alpha)(1-\lambda+n\lambda)z_0^{n-1}},$$

$$n = 2, 3...$$

where $k \geq 1$. Then $f \in TC_sR_\lambda(\alpha, \mu, z_0)$, if and only if it can be expressed in the form

$$f(z) = \sum_{n=2}^{\infty} d_nf_n(z), where \;\; d_n \geq 0 \;\; and \;\; \sum_{n=2}^{\infty} d_n = 1$$

Theorem 4.4. Let $f \in TC_sR_\lambda(\alpha, \mu, z_0)$.. Then f is convex in the disc $|z| < r_\lambda(\alpha, \mu)$, where

$$r_\lambda(\alpha, \mu) = \inf_n \left\{ \frac{n(1+\alpha) - \alpha(1-(-1)^n)}{n^2(1-\alpha)} A_n(\mu) \right\}^{\frac{1}{n-1}} \quad n = 2, 3.....$$

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NUMERICAL SOLUTION OF TRANSIENT MHD FREE CONVECTION FLOW OF AN INCOMPRESSIBLE VISCOUS FLUID THROUGH POROUS MEDIUM ALONG AN INCLINED PLATE WITH OHMIC DISSIPATION

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Abstract

The present paper is to investigate the effect of porous medium on transient free convective flow of a viscous incompressible electrically conducting fluid along an inclined isothermal non-conducting plate in the presence of transverse magnetic field, viscous dissipation and Ohmic dissipation. The governing equations of continuity, momentum and energy are solved using explicit finite difference scheme. The velocity and temperature distributions are discussed numerically and presented through graphs and Tables. Skin-friction coefficient and the Nusselt number at the plate are derived, discussed and their numerical values for various values of physical parameters are presented through tables.

1 Introduction

Thermal boundary layer flow problems are classified into categories e.g. (i) free natural convection flow and (ii) forced convection flow and have many applications in the area of industries and engineering.

The study of natural convection flow (Schlichting and Gersten 1999; Bansal 1977) finds its applications in nuclear reactor, spacecraft design, chemical industry etc. the unsteady MHD free convective flows of dissipative fluid are important because of non-linearity of the governing equations.

Keywords and phrases : Transient free convection, magnetic field, porous medium, viscous dissipation, Ohmic heating, finite difference technique.

AMS Subject Classification : 70D05, 76W05.

The problem of free convection flow past a prous/non-porous vertical plate has been considered by many researchers, e.g. Kumar and Yadav (2007) have studied unsteady MHD free convection flow through porous medium with heat and mass transfer past a porous vertical moving plate with heat Source/sink. Mittal et. al. (2010) have studied on the vorticity of unsteady MHD free convection flow through porous medium with heat and mass transfer past a porous vertical moving plate with heat source/sink.

Seigal (Seigal 1958) solved the problem of unsteady free convective flow along vertical plate using integral method. Raptis and Tzivanidis (Raptis and Tzivanidis 1981) obtained the numerical solutions of unsteady flow along accelerated vertical plate and unsteady MHD with constant heat flux and presented exact solution. Soundalgekar et al. (soundalgekar et al. 1977) considered the transient free convection of incompressible dissipative fluid on vertical plate. Muthucumaraswamy (Muthucumaraswamy 2003) studied unsteady flow along accelerated plate with mass diffusion. MHD free convective flow of dissipative fluid on the vertical plate has been discussed by Sridhar et al. (Sridhar et al. 2006). Recently, Sharma and Singh (2009) have discussed on Numerical solution of transient MHD convection flow of an incompressible viscous fluid along an inclined plate with Ohmic dissipation.

The aim of present paper is to investigate unsteady natural convection in the boundary layer in a viscous incompressible electrically conducting dissipative fluid through porous medium along an inclined isothermal non-conduction plate considering the Ohmic heating in the presence of transverse magnetic field. The problem is coupled non-linear partial equation whose exact solution is not possible; hence finite difference technique is employed to obtain effects of physical parameter on velocity and temperature profiles.

2 Formulation of the Problem

Consider unsteady laminar two-dimensional free convective flow of a viscous incompressible electrically conducting fluid through porous medium along an inclined non-conducting plate and y^*-axis is normal to the plate. Magnetic field of uniform intensity B_0 is applied in y^*- direction. Initially, the temperature of fluid and plate are assumed to be same and for $t^* > 0$, the plate temperature is raised to T_w. While formulation of the problem, it is assumed that the external field is zero, also electrical field due to polarization of charges and Hall Effect are neglected. Incorporating the Boussinesq approximation within the boundary layer, the governing equations of momentum and energy are function of y^* and t^* in the presence of transverse magnetic field (Bansal 1944) are given below

$$\frac{\partial u^*}{\partial t^*} = v\frac{\partial^2 u^*}{\partial y^{*2}} + g\beta(T^* - T_\infty)\cos\gamma - \frac{\sigma B_0^2}{\rho}u^* - \frac{v}{K^*}u^* \tag{1}$$

$$\rho C_p\frac{\partial T^*}{\partial t^*} = k\frac{\partial^2 T^*}{\partial y^{*2}} + \mu(\frac{\partial u^*}{\partial y^*}) + \sigma B_0^2 u^{*2} \tag{2}$$

Where u^* is velocity of fluid in x^* direction, g is acceleration due to gravity of the Earth, β is coefficient of thermal expansion, γ is inclination angle form the vertical direction, ρ is density of fluid, C_p specific heat at constant pressure, v the kinematic viscosity, k the thermal conductivity, T^* is temperature of fluid and T_∞ is the temperature of the far away from the plate.

The initial and boundary conditions are

$$\left.\begin{array}{llll} t^* \leq 0, & u^* = 0, & T^* = T_\infty & \text{for all } y^* \\ t^* > 0, & u^* = 0, & T^* = T_w & \text{at } y^* = 0 \\ & u^* = 0, & T^* = T_\infty & \text{as } y^* \to \infty \end{array}\right\} \quad (3)$$

Introducing the following non-dimensional quantities

$$y = \left\{\frac{g\beta(T_w - T_\infty)}{V^2}\right\}^{\frac{1}{3}} y^*,$$

$$t = \left\{\frac{g^2\beta^2(T_w - T_\infty)^2}{V}\right\}^{\frac{1}{3}} t^*, \qquad u = \{vg\beta(T_w - T_\infty)\}^{-\frac{1}{3}} u^*,$$

$$\theta = \frac{T^* - T_\infty}{T_w - T_\infty}$$

and $Pr = \dfrac{\mu C_p}{k}$ into the equations (1) and (2), we get

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \theta \cos\gamma - \left(M + \frac{1}{K}\right) u \quad (4)$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial y^2} + Ec\left(\frac{\partial u}{\partial y}\right) + EcMu^2, \quad (5)$$

Where $\theta = \left\{\dfrac{T^* - T_\infty}{T_w - T_\infty}\right\}$ is dimensionless temperature

$M = \left\{\dfrac{v\sigma B_0^2}{u_0^2 \rho}\right\}$ is magnetic parameter

$K^* = \dfrac{Kv^2}{u_0^2}$ is the porosity parameter

$Ec = \left\{\dfrac{u_0^2}{C_p(T_w - T_\infty)}\right\}$ is Eckert number

$u_0 = \{vg\beta(T_w - T_\infty)\}^{\frac{1}{3}}$ and Pr is Prandtl number.

$$\left.\begin{array}{llll} t \leq 0, & u = 0, & \theta = 0 & \text{for all } y \\ t > 0, & u = 0, & \theta = 1 & \text{at } y = 0 \\ & u = 0, & \theta \to 0 & \text{as } y \to \infty \end{array}\right\} \quad (6)$$

3 Method of Solution

The equations (4) and (5) are coupled differential equations; therefore exact solution is not possible. Hence explicit finite difference method (Jain 2000;) Muralidhar and Sundararajan 2003) is employed to seek the solution of the equations (4) and (5) under the boundary conditions (6). The finite difference equations (4) and (5) are as follows

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta y)^2} + \theta_{i,j}\cos\gamma + \left(M + \frac{1}{K}\right)u_{i,j}, \tag{7}$$

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = \frac{1}{Pr}\frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta y)^2} + Ec\left(\frac{u_{i+1,j} - u_{i,j}}{\Delta y}\right) + MEc(u_{i,j})^2 \tag{8}$$

Where index i refers to y and j to time t, and during computation $\Delta y = 0.1$ and $\Delta t = 0.00125$. The scheme is found to be stable and convergent while checking for different values Δt of and no significant change was observes.

4 Skin - Friction

The skin-friction coefficient at the plate in non-dimensional form is given by

$$C_f = \frac{\tau}{\mu u_0/L} = \left(\frac{du}{dy}\right)_{y=0}, \tag{9}$$

Where

$$\tau = \mu\left(\frac{\partial u}{\partial y}\right)_{y^*=0}$$

5 Rate of Heat Transfer

The rate heat transfer at the plate in the form of Nusselt number is given by

$$Nu = \frac{qL}{k(T_w - T_\infty)} = -\left(\frac{d\theta}{dy}\right)_{y=0}, \tag{10}$$

Where

$$q = -k\left(\frac{\partial T}{\partial y}\right)_{y^*=0}$$

The values of C_f and Nu at the plate are evaluated using Newton's interpolation formula.

6 Results and Discussion

Numerical calculations have been carried out for dimensionless velocity of fluid (u) and temperature profiles θ for different values of parameters and are displayed in Figures-(1) to (7).

Figure - (1) depicts that with the increase in magnetic field intensity, the fluid velocity decreases. This agrees with the natural phenomena because in the presence of transverse magnetic field, Lorentz force sets in, which impedes the fluid velocity.

Figure - (2) depicts that with increase the permeability of porous medium, the fluid velocity increases due to Darcy's law.

Figure - (3) depicts that the increase in angle of inclination reduces fluid velocity because; increase in angle (γ) reduces buoyancy forces.

Figures - (4) and (5) depict that with increase in Prandtl number, fluid velocity and fluid temperature decrease. The boundary layer and thermal boundary layer thicknesses reduce with increase in the Prandtl number.

It is seen from figures - (6) and (7) that fluid velocity and fluid temperature increases with the lapse of time. The boundary layer and thermal boundary layer thicknesses increase with the time.

It is seen from table - (1) that velocity distribution of fluid, dust particles and temperature distribution for Ec. The increase in the viscous dissipative heat leads to increase in velocity distribution of fluid, dust particles and temperature distribution.

It is observed form table - (2) that with the increase in the value of magnetic field intensity and porosity, fluid temperature increased. But increase in angle of inclination reduces fluid temperature.

It is seen from table - (3) that skin-friction coefficient decreases and Nusselt number increases with the increase in the magnetic field intensity, angle of inclination or Prandtl number. The increase in the viscous dissipative heat and porosity leads to increase in skin-friction and decrease in the Nusselt number. As the time increases, skin-friction increases and rate of heat transfer decreases.

Table- (1) Numerical values of Velocity distribution of fluid and temperature distribution for different values of Ec at $M = 0.5, K = 2, \gamma = 0^o, Pr = 0.71$ & $t = 0.1$.

y	Velocity distribution of fluid (u)		Temperature distribution of fluid (θ)	
	$Ec = 0.3$	$Ec = 3.0$	$Ec = 0.3$	$Ec = 3.0$
0.0	0.0000000	0.0000000	1.0000000	1.0000000
0.1	0.0142652	0.0142662	0.8500521	0.8501312
0.2	0.0210752	0.0210769	0.7053034	0.7053852
0.3	0.0227342	0.0227363	0.5704130	0.5705028
0.4	0.0211911	0.0211935	0.4490045	0.4491069
0.5	0.0179715	0.0179738	0.3433617	0.3434678
0.6	0.0141621	0.0141641	0.2543061	0.2544029
0.7	0.0104448	0.0104464	0.1812493	0.1813278
0.8	0.0071632	0.0071644	0.1223794	0.1224368
0.9	0.0044048	0.0044055	0.0749224	0.0749600
1	0.0020815	0.0020818	0.0354223	0.0354418

Table- (2) Numerical values of temperature distribution for different values of M, K and γ at $Ec = 0.3, Pr = 0.71 and dt = 0.1$.

Y	$M=0.5, K=2$ $\gamma=0^\circ$	$M=10, K=2$ $\gamma=0^\circ$	$M=0.5, K=10$ $\gamma=0^\circ$	$M=0.5, K=2$ $\gamma=45^\circ$
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	0.85005205	0.85005642	0.85005225	0.85004765
0.2	0.70530335	0.70531412	0.70530353	0.70529881
0.3	0.57041301	0.57042647	0.57041318	0.57040802
0.4	0.44900454	0.44901686	0.44900474	0.44899886
0.5	0.34336172	0.34337096	0.34336193	0.34335583
0.6	0.25430608	0.25431205	0.25430628	0.25430070
0.7	0.18124928	0.18125268	0.18124946	0.18124492
0.8	0.12237938	0.12238105	0.12237951	0.12237619
0.9	0.07492239	0.07492303	0.07492248	0.07492030
1	0.03542233	0.03542245	0.03542237	0.03542125

Table- (3) Numerical values of skin friction coefficient and Nusselt number for different values of M, K, Pr, γ, t and Ec.

Pr	γ	M	K	Ec	t	C_f	Nu
0.71	0°	0.5	2	0.3	0.1	0.18828026	1.50822077
0.71	0°	5	2	0.3	0.1	0.17448563	1.50822086
0.71	0°	10	2	0.3	0.1	0.16165149	1.50822466
0.71	30°	0.5	2	0.3	0.1	0.16305517	1.50827419
0.71	30°	5	2	0.3	0.1	0.15110858	1.50827425
0.71	30°	10	2	0.3	0.1	0.13999384	1.50827710
0.71	45°	0.5	2	0.3	0.1	0.13313373	1.50832760
0.71	45°	5	2	0.3	0.1	0.12337930	1.50832764
0.71	45°	10	2	0.3	0.1	0.11430411	1.50832954
1.2	30°	0.5	2	0.3	0.1	0.14346648	1.96220964
1.2	30°	5	2	0.3	0.1	0.13412413	1.96221378
1.2	30°	10	2	0.3	0.1	0.12532919	1.96221883
2.1	30°	0.5	2	0.3	0.1	0.12268656	2.60122297
2.1	30°	5	2	0.3	0.1	0.11575942	2.60122974
2.1	30°	10	2	0.3	0.1	0.10915619	2.60123618
0.71	0°	0.5	5	0.3	0.1	0.18928890	1.50821732
0.71	0°	0.5	10	0.3	0.1	0.18962780	1.50821615
0.71	0°	0.5	2	1.5	0.1	0.18828609	1.50736611
0.71	0°	0.5	2	3	0.1	0.18829338	1.50629762
0.71	0°	0.5	2	0.3	0.1	0.18828026	1.50822077
0.71	0°	0.5	2	0.3	0.11	0.19728132	1.43818307
0.71	0°	0.5	2	0.3	0.12	0.20580553	1.37756619

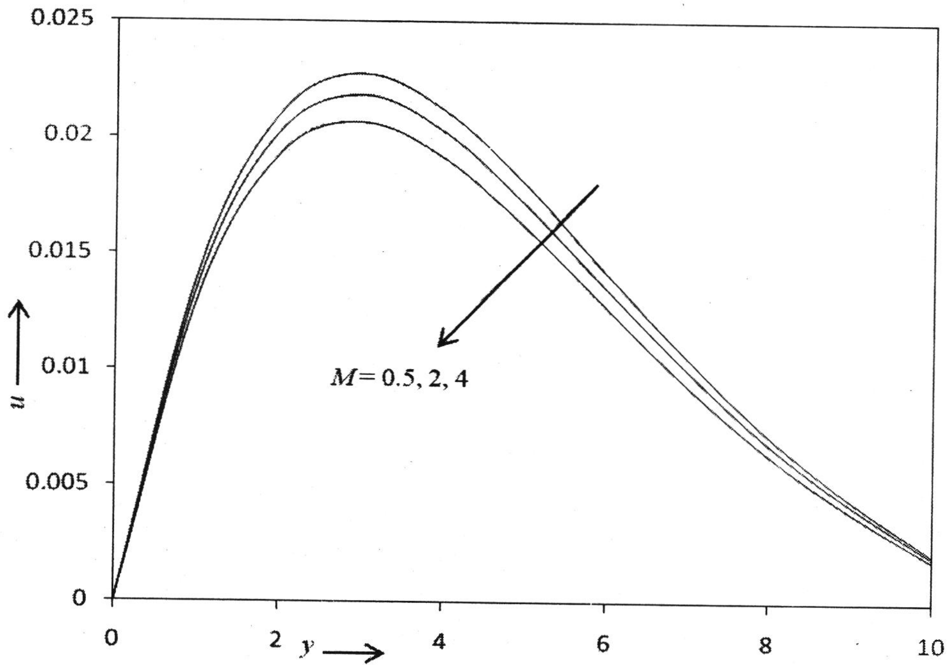

Fig. - 1: Velocity profile of fluid for different values of M.

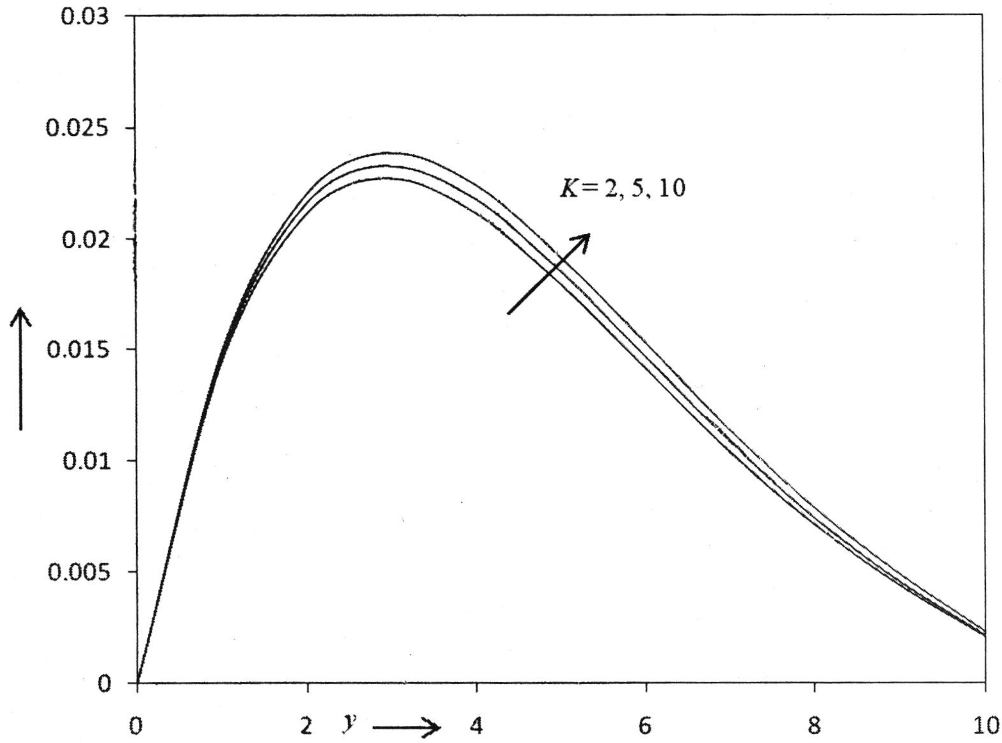

Fig. - 2: Velocity profile of fluid for different values of K.

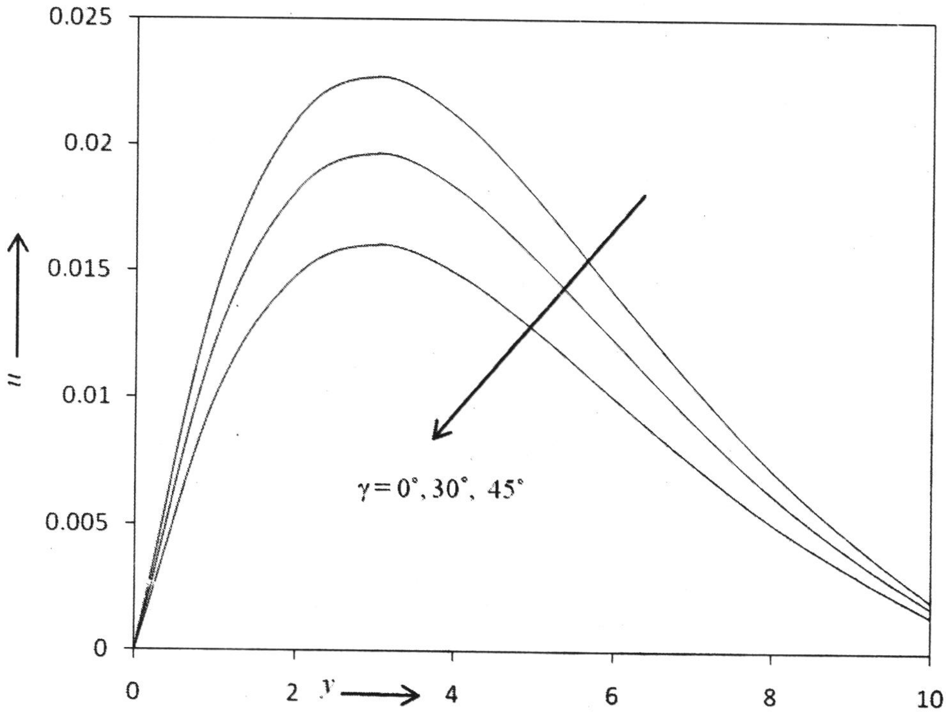

Fig. - 3: Velocity profile of fluid for different values of γ.

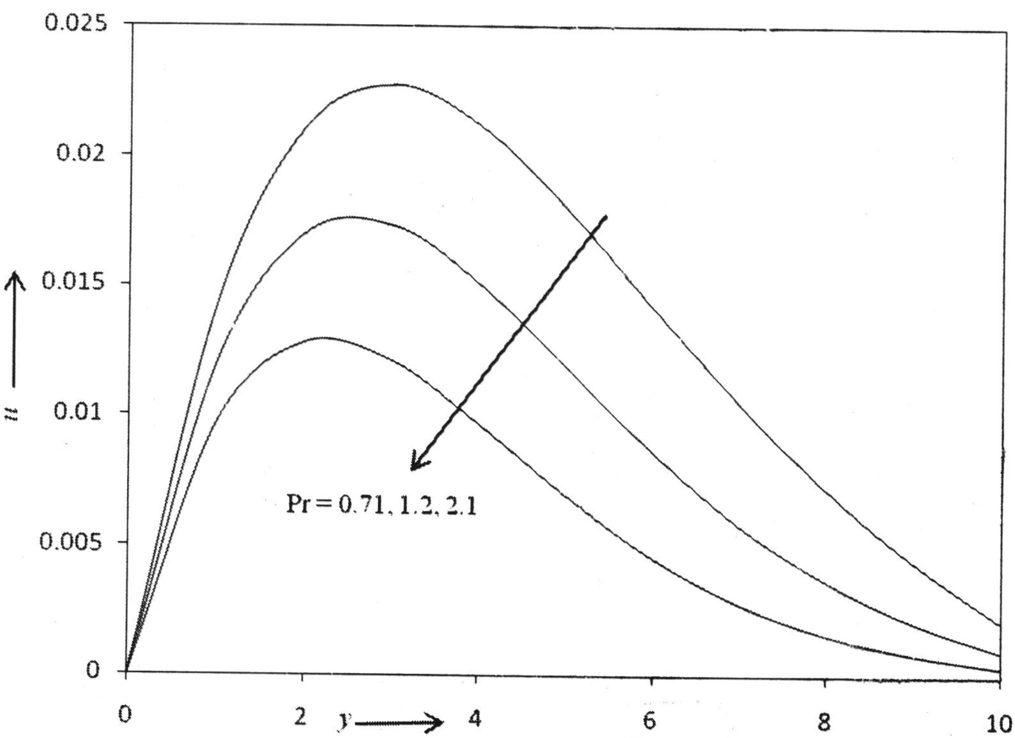

Fig. - 4: Velocity profile of fluid for different values of Pr.

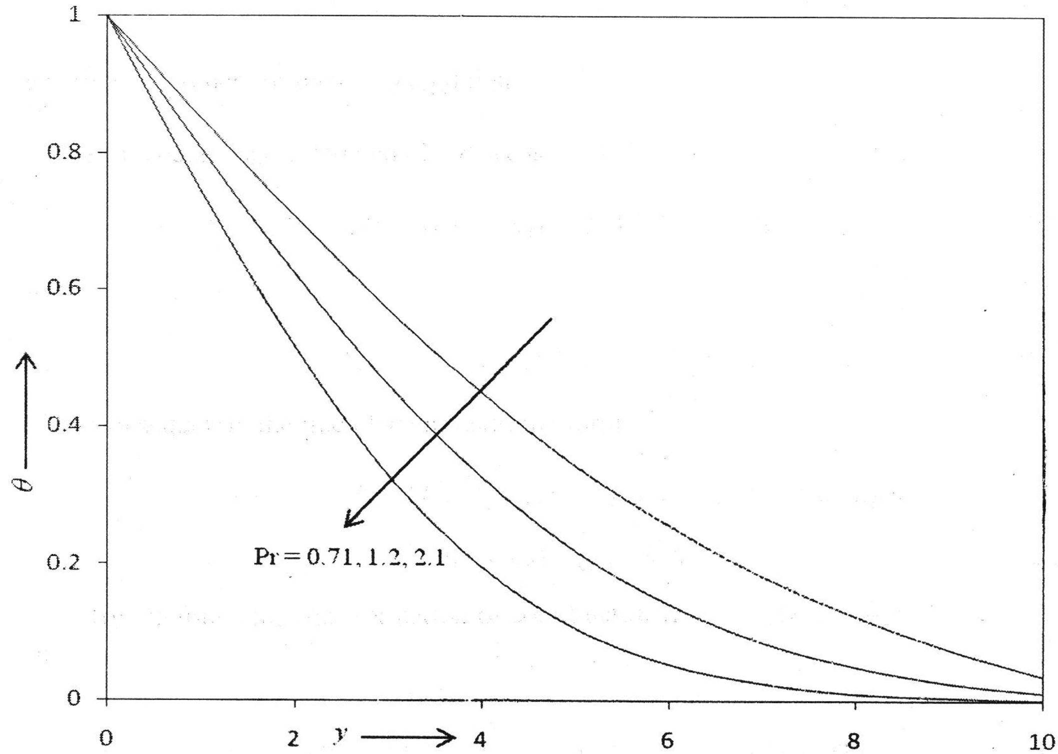

Fig. - 5: Temperature profile of fluid for different values of Pr.

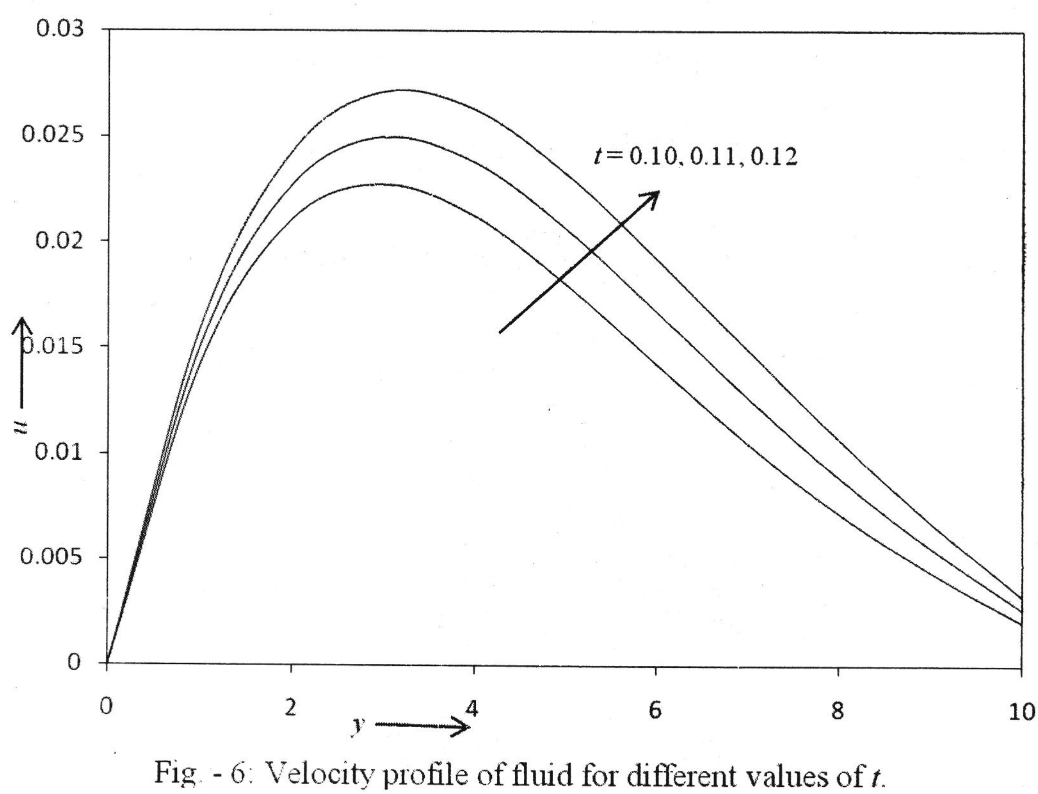

Fig. - 6: Velocity profile of fluid for different values of t.

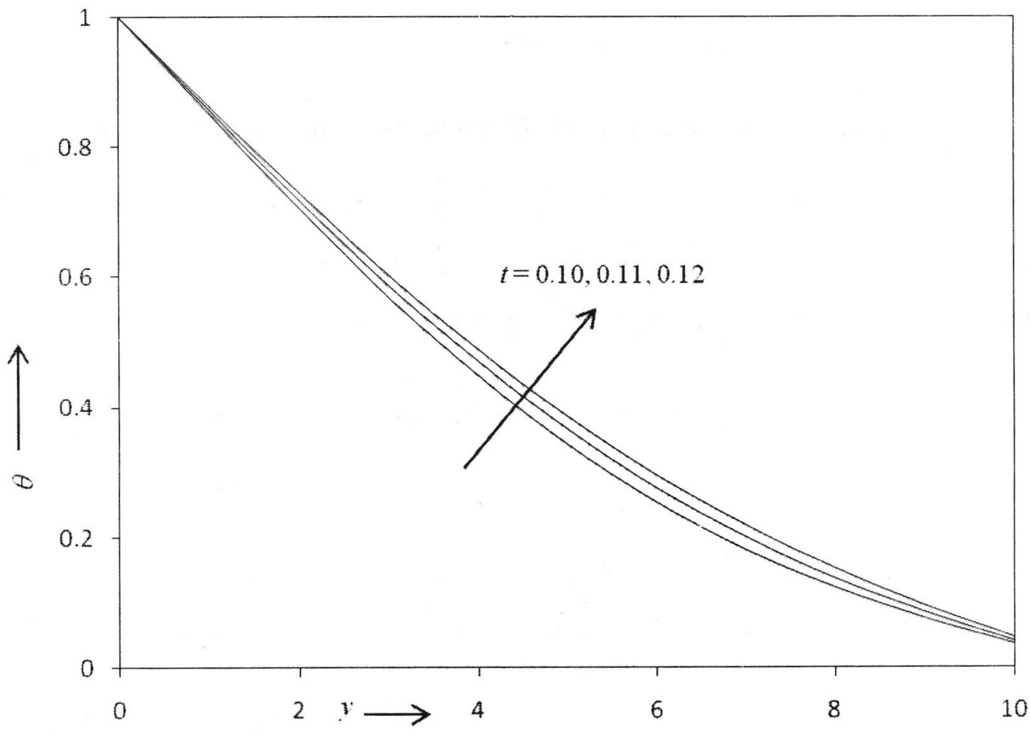

Fig. - 7: Temperature profile of fluid for different values of t.

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ON GENERALIZED COUPLED FIXED POINT RESULTS IN PARTIAL METRIC SPACES

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Abstract

In this work, we introduce the concept of n-tuple fixed point which is a generalization of coupled fixed point for mappings in complete partial metric spaces and obtain existence and uniqueness theorems for different contractive conditions. Also, we give a very important comment that any n-tuple fixed point of $F : X^n = X \times X \times X ... \times X \to X$ if and only if is a fixed point of $G : X^n \to X^n$, where $(X^n, \hat{p})$ is a partial metric space induced by a partial metric space (X, p). Our results generalize relevant results due to Bhaskar and Lakshmikantham [4], Borcut, Berinde [3] and Hassen Aydi [2].

1 Introduction and preliminaries

In 2006, T. G. Bhaskar and V. Lakshmikantham [4] given the notion of coupled fixed point and proved some interesting coupled fixed point theorems for mapping satisfying a mixed monotone property. M. Borcut, V. Berinde [3] introduced the concept of tripled fixed point for nonlinear contractive mappings of the form $F : X \times X \times X \to X$, and obtained existence and uniqueness theorems in partially ordered complete metric spaces X. coupled common fixed point results and coupled coincidence point results existing in literature, e.g., [1, 5, 9, 10].

In a recent paper, Hassen Aydi [2] introduced some coupled fixed point results for mappings satisfying different contractive conditions on complete partial metric spaces.

Keywords and phrases : Partial metric space, Contractive type operator, N-tuples fixed point, Existence and uniqueness.

AMS Subject Classification : Primary 54H25; Secondary 47H10.

The purpose of this paper is to present some n-tuple fixed point theorems for different contractive mappings and we prove that a point $(x_1, x_2, ..., x_n)$ is n-tuple fixed point of $F : X^n \to X$ if and only if $(x_1, x_2, ..., x_n)$ is a fixed point of $G : X^n \to X^n$, where $(X^n, \hat{p})$ is a partial metric space induced by a partial metric space (X, p) as follows:

$$\hat{p}((x^1, x^2, ..., x^n), (y^1, y^2, ..., y^n)) = \sum_{i=1}^{n} j_i p(x^i, y^i), \quad \sum_{i=1}^{n} j_i < 1, \tag{1}$$

where $x^1, x^2, ..., x^n, y^1, y^2, ..., y^n \in X$, $j_1, j_2, ..., j_n$ are nonnegative constants.

Now, we present some basic notions and results due to T. G. Bhaskar and V. Lakshmikantham [4], M. Borcut and V. Berinde[3].

Definition 1.1 [4]. Call an element $(x, y) \in X \times X$ a coupled fixed point of the mapping F if

$$F(x, y) = x, F(y, x) = y.$$

Theorem 1.1 [4]. Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a constant $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)] \ \forall \ x \geq u, \ \ y \leq v. \tag{2}$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \ and \ y_0 \geq F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$x = F(x, y) \ and \ y = F(y, x).$$

Definition 1.2 [3]. An element $(x, y, z) \in X \times X \times X$ is said to be a tripled fixed point of the mapping F if

$$F(x, y, z) = x, F(y, x, z) = y \ and \ F(z, y, x) = z.$$

Theorem 1.2 [3]. Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist constants $j, k, l \in [0, 1)$ with , $j + k + l < 1$ for which

$$d(F(x, y, z), F(u, v, w))$$

$$\leq jd(x, u) + kd(y, v) + ld(z, w) \ \forall \ x \geq u, \ \ y \leq v, \ \ z \geq w. \tag{3}$$

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \ \ y_0 \geq F(y_0, x_0, z_0) \ and \ z_0 \leq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$x = F(x, y, z), \quad y = F(y, x, z) \ \ and \ \ z = F(z, y, x).$$

In 1994, S. G. Matthews [6] introduced the notion of a partial metric space as a generalization of metrics where self-distances are not necessarily zero and obtained a Banach contraction mapping for these spaces. First, we summarize in the following the basic notions and results established in partial metric spaces.

Definition 1.3 ([8, 7]). Let X be a nonempty set. A function $p : X \times X \to [0, \infty)$ is called a distance on X. The pair (X, p) is called a partial metric space if p satisfies the following conditions:

(p_1) $p(x, x) = p(x, y) = d(y, y) \Longleftrightarrow x = y$,
(p_2) $p(x, x) \leq p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$,
for all $x, y, z \in X$.

Remark 1.1. We note that if (X, p) be a partial metric space then $p(x, y) = 0 \Rightarrow x = y$ but the converse my not be true.

If (X, p) a partial metric space, then the function $\widetilde{d} : X \times X \to [0, \infty)$ given by

$$\widetilde{d}(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{4}$$

is a metric on X.

Definition 1.4 ([6, 8]). Let (X, p) be a partial metric space. Then,

(i) a sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim\limits_{n \to \infty} p(x, x_n) = \lim\limits_{n \to \infty} p(x_n, x_n)$;

(ii) a sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if there exist $a \geq 0$ such that for each $\epsilon > 0$ there exist k such that for all $n, m > k$, $| \, p(x_n, x_m) - a \, | < \epsilon$.

(iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point in X, such that $p(x, x) = \lim\limits_{n,m \to \infty} p(x_n, x_m)$.

Lemma 1.1 ([6, 7]). Let (X, p) be a partial metric space;

(1) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space $(X, \widetilde{d})$.

(2) a partial metric space (X, p) is complete if and only if the metric space $(X, \widetilde{d})$ is complete; furthermore, $\lim\limits_{n \to \infty} \widetilde{d}(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n \to +\infty} p(x_n, x) = \lim_{n,m \to +\infty} p(x_n, x_m). \tag{5}$$

Theorem 1.3 ([2]). Let (X,p) be a complete partial metric space. Suppose that the mapping $F : X \times X \to X$ satisfy the following contractive condition for all $x, y, u, v \in X$.

$$p(F(x,y), F(u,v)) \leq kp(x,u) + lp(y,v)] \tag{6}$$

where k, l are nonnegative constants with $k + l < 1$. Then F has a unique coupled fixed point.

2 N-tuple fixed point theorems on partial metric spaces

Throughcut of this section, X or (X,p) will denote a partial metric space.

Definition 2.1. An element $(x^1, x^2, ..., x^n) \in X^n$ is called a n-tuple fixed point of $F : X^n \to X$ if $F(x^1, x^2, ..., x^n) = x^1, F(x^2, x^1, ..., x^n) = x^2, \cdots, F(x^n, x^{n-1}, ..., x^1) = x^n$.

Remark 2.1. We note that in the Definition 2.1, if $n = 2$ then F has a coupled point and if $n = 3$ then F has a tripled point.

Theorem 2.1. Let (X,p) be a complete partial metric space. Suppose $F : X^n \to X$ satisfies the following contractive condition for all $x^1, x^2, ..., x^n$, $y^1, y^2, ..., y^n \in X$

$$p(F(x^1, x^2, ..., x^n), F(y^1, y^2, ..., y^n)) \leq \sum_{i=1}^{n} j_i p(x^i, y^i), \tag{7}$$

where $j_1, j_2, ..., j_n$ are nonnegative constants with $\sum_{i=1}^{n} j_i < 1$. Then F has a unique n-tuple fixed point.

Proof. Choose $x_0^1, x_0^2, ..., x_0^n \in X$ and set $x_1^1 = F(x_0^1, x_0^2, ..., x_0^n), x_1^2 = F(x_0^2, x_0^1, ..., x_0^n), ...$ and $x_1^n = F(x_0^n, x_0^{n-1}, ..., x_0^1)$. Repeating this process, set $x_{m+1}^1 = F(x_m^1, x_m^2, ..., x_m^n)$, $x_{m+1}^2 = F(x_m^2, x_m^1, ..., x_m^n), ...$ and $x_{m+1}^n = F(x_m^n, x_m^{n-1}, ..., x_m^1)$. Then by (7) we have

$$\begin{aligned} p(x_m^1, x_{m+1}^1) &= p(F(x_{m-1}^1, x_{m-1}^2, ..., x_{m-1}^n), \\ &\quad F(x_m^1, x_m^2, ..., x_m^n)) \\ &\leq j_1 p(x_{m-1}^1, x_m^1) + j_2 p(x_{m-1}^2, x_m^2) + ... + \\ &\quad j_n p(x_{m-1}^n, x_m^n). \end{aligned} \tag{8}$$

Similarly, we obtain

$$\begin{aligned} p(x_m^2, x_{m+1}^2) &= p(F(x_{m-1}^2, x_{m-1}^1, ..., x_{m-1}^n), \\ &\quad F(x_m^2, x_m^1, ..., x_m^n)) \\ &\leq j_1 p(x_{m-1}^2, x_m^2) + j_2 p(x_{m-1}^1, x_m^1) + ... + \\ &\quad j_n p(x_{m-1}^n, x_m^n), \end{aligned}$$

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$$\begin{aligned} p(x_m^n, x_{m+1}^n) &= p(F(x_{m-1}^n, x_{m-1}^{n-1}, ..., x_{m-1}^1), \\ &\quad F(x_m^n, x_m^{n-1}, ..., x_m^1)) \\ &\leq j_1 p(x_{m-1}^n, x_m^n) + j_2 p(x_{m-1}^{n-1}, x_m^{n-1}) + ... + \\ &\quad j_n p(x_{m-1}^1, x_m^1) \end{aligned} \tag{9}$$

Therefore, by letting

$$d_m = p(x_m^1, x_{m+1}^1) + p(x_m^2, x_{m+1}^2) + ... + p(x_m^n, x_{m+1}^n) \tag{10}$$

we have

$$\begin{aligned} d_m &= p(x_m^1, x_{m+1}^1) + p(x_m^2, x_{m+1}^2) + ... + p(x_m^n, x_{m+1}^n) \\ &\leq (j_1 + j_2 + ... + j_n) \left[p(x_{m-1}^1, x_m^1) + p(x_{m-1}^2, x_m^2) \right. \\ &\quad \left. + ... + p(x_{m-1}^n, x_m^n)\right] \\ &= (j_1 + j_2 + ... + j_n) d_{m-1}. \end{aligned}$$

Consequently, if we set $\delta = j_1 + j_2 + ... + j_n$, then, for each $n \in N$(the set of all natural number) we have

$$d_m \leq \delta d_{m-1} \leq \delta^2 d_{m-2} \leq ... \leq \delta^m d_0. \tag{11}$$

If $d_0 = 0$ then $p(x_0^1, x_1^1) + p(x_0^2, x_1^2) + ... + p(x_0^n, x_1^n) = 0$. Hence from Remark 1.1, we get that $x_0^1 = x_1^1 = F(x_0^1, x_0^2, ..., x_0^n)$, $x_0^2 = x_1^2 = F(x_0^2, x_0^1, ..., x_0^n)$, ... and $x_0^n = x_1^n = F(x_0^n, x_0^{n-1}, ..., x_0^1)$, meaning that $(x_0^1, x_0^2, ..., x_0^n)$ is a n-tuple fixed point of F. Now, let $d_0 > 0$. for each $k \geq m$, we have, in view of the condition (p4)

$$\begin{aligned} p(x_k^1, x_m^1) &\leq p(x_k^1, x_{k-1}^1) + p(x_{k-1}^1, x_{k-2}^1) - p(x_{k-1}^1, x_{k-1}^1) \\ &\quad + p(x_{k-2}^1, x_{k-3}^1) + p(x_{k-3}^1, x_{k-4}^1) - p(x_{k-3}^1, x_{k-3}^1) \\ &\quad + ... + p(x_{m+2}^1, x_{m+1}^1) + p(x_{m+1}^1, x_m^1) - p(x_{m+1}^1, x_{m+1}^1) \\ &\leq p(x_k^1, x_{k-1}^1) + p(x_{k-1}^1, x_{k-2}^1) + ... + p(x_{m+1}^1, x_m^1). \end{aligned}$$

Similarly we obtain

$$p(x_k^2, x_m^2) \le p(x_k^2, x_{k-1}^2) + p(x_{k-1}^2, x_{k-2}^2) + ... + p(x_{m+1}^2, x_m^2),$$

$$\cdot$$

$$\cdot$$

$$p(x_k^n, x_m^n) \le p(x_k^n, x_{k-1}^n) + p(x_{k-1}^n, x_{k-2}^n) + ... + p(x_{m+1}^n, x_m^n).$$

Thus

$$\begin{aligned} p(x_k^1, x_m^1) + p(x_k^2, x_m^2) + ... + p(x_k^n, x_m^n) &\le (d_{k-1} + d_{k-2} + ... + d_m) \\ &\le (\delta^{k-1} + \delta^{k-2} + ... + \delta^m) d_0 \\ &\le \frac{\delta^m}{1-\delta} d_0. \end{aligned} \tag{12}$$

By definition of $\widetilde{d}$, we have $\widetilde{d} \le 2p(x, y)$, so, for any $k \ge m$

$$\begin{aligned} \widetilde{d}(x_k^1, x_m^1) + \widetilde{d}(x_k^2, x_m^2) + ... + \widetilde{d}(x_k^n, x_m^n) &\le 2p(x_k^1, x_m^1) + 2p(x_k^2, x_m^2) \\ &\quad + ... + 2p(x_k^n, x_m^n) \\ &\le 2\frac{\delta^m}{1-\delta} d_0. \end{aligned} \tag{13}$$

which implies that $\{x_m^1\}, \{x_m^2\}, ...$ and $\{x_m^n\}$ are Cauchy sequences in $(X, \widetilde{d})$ because of $0 \le \delta = j_1 + j_2 + ... + j_n < 1$. Since the partial metric space (X, p) is complete, hence by Lemma 1.1, the metric space $(X, \widetilde{d})$ is complete, so there exist $u^1, u^2, ..., u^n \in X$ such that

$$\lim_{m\to\infty} \widetilde{d}(x_m^1, u^1) = \lim_{m\to\infty} \widetilde{d}(x_m^2, u^2) = ... = \lim_{m\to\infty} \widetilde{d}(x_m^n, u^n) = 0 \tag{14}$$

From Lemma 1.1, we get

$$\begin{aligned} p(u^1, u^1) &= \lim_{m\to\infty} p(x_m^1, u^1) = \lim_{m\to\infty} p(x_m^1, x_m^1), \\ p(u^2, u^2) &= \lim_{m\to\infty} p(x_m^2, u^2) = \lim_{m\to\infty} p(x_m^2, x_m^2), \\ &\cdot \\ &\cdot \\ &\cdot \\ p(u^n, u^n) &= \lim_{m\to\infty} p(x_m^n, u^n) = \lim_{m\to\infty} p(x_m^n, x_m^n). \end{aligned} \tag{15}$$

By condition $(p2)$ and (11) we have

$$p(x_m^1, x_m^1) \le p(x_m^1, x_{m+1}^1) \le d_m \le \delta^m d_0, \tag{16}$$

which show that $\lim_{m\to+\infty} p(x_m^1, x_m^1) = 0$. It follows that

$$p(u^1, u^1) = \lim_{m\to\infty} p(x_m^1, u^1) = \lim_{m\to\infty} p(x_m^1, x_m^1) = 0. \tag{17}$$

Similarly, one has

$$p(u^2,u^2) = \lim_{m\to\infty} p(x_m^2,u^2) = \lim_{m\to\infty} p(x_m^2,x_m^2) = 0,$$

$$\cdot$$

$$\cdot \qquad (18)$$

$$\cdot$$

$$p(u^n,u^n) = \lim_{m\to\infty} p(x_m^n,u^n) = \lim_{m\to\infty} p(x_m^n,x_m^n) = 0.$$

By using the contractivity condition on F one obtain

$$\begin{aligned} p(F(u^1,u^2,...,u^n),u^1) &\leq p(F(u^1,u^2,...,u^n),x_{m+1}^1) \\ &+p(x_{m+1}^1,u^1) - p(x_{m+1}^1,x_{m+1}^1) \\ &\leq p(F(u^1,u^2,...,u^n), \\ &F(x_m^1,x_m^2,...,x_m^n)) + p(x_{m+1}^1,u^1) \\ &\leq j_1 p(x_m^1,u^1) + j_2 p(x_m^2,u^2) \\ &+... + j_n p(x_m^n,u^n) + p(x_{m+1}^1,u^1), \end{aligned} \qquad (19)$$

and letting $m \to +\infty$, then from (17) and (18), we obtain $p(F(u^1,u^2,...,u^n),u^1) = 0$, so $F(u^1,u^2,...,u^n) = u^1$. Similarly, we have $F(u^2,u^1,...,u^n) = u^2$, $F(u^3,u^2,...,u^n) = u^3$, ... and $F(u^n,u^{n-1},...,u^1) = u^n$, meaning that $(u^1,u^2,...,u^n)$ is n-tuple fixed point of F.

Now if $(v^1,v^2,...,v^n)$ is another n-tuple fixed point of F, then

$$\begin{aligned} p(u^1,v^1) &= p(F(u^1,u^2,...,u^n),F(v^1,v^2,...,v^n)) \\ &\leq j_1 p(u^1,v^1) + ... + j_n p(u^n,v^n) \\ p(u^2,v^2) &= p(F(u^2,u^1,...,u^n),F(v^2,v^1,...,v^n)) \\ &\leq j_1 p(u^2,v^2) + ... + j_n p(u^n,v^n) \\ &\cdot \\ &\cdot \\ &\cdot \\ p(u^n,v^n) &= p(F(u^n,u^{n-1},...,u^1),F(v^n,v^{n-1},...,v^1)) \\ &\leq j_1 p(u^n,v^n)) + ... + j_n p(u^1,v^1) \end{aligned} \qquad (20)$$

It follows that

$$\begin{aligned} &p(u^1,v^1) + p(u^2,v^2) + ... + p(u^n,v^n) \leq (j_1 + j_2 + ... + j_n) \\ &[p(u^1,v^1) + ... + p(u^n,v^n)] \end{aligned}$$

In view of $(j_1+j_2+...+j_n) < 1$, this implies that $p(u^1,v^1)+p(u^2,v^2)+...+p(u^n,v^n) = 0$, so $u^1 = v^1$, ...and $u^n = v^n$. The proof of Theorem 2.1 is completed.

Remark 2.2. Theorem 2.1 extends the Theorem 1.3 (Theorem 2.1 of [2]) for $n = 2$.

If we put $j_1 = j_2 = ... = j_n = \frac{k}{2}$, in Theorem 2.1, we have the following corollary

Corollary 2.1. Let (X,p) be a partial a complete metric space. Suppose $F : X^n \to X$ satisfies the following contractive condition for all $x^1, x^2, ..., x^n, y^1, y^2, ..., y^n \in X$

$$\begin{aligned} & p(F(x^1, x^2, ..., x^n), F(y^1, y^2, ..., y^n)) \\ \leq \; & \tfrac{k}{2}[p(x^1, y^1) + p(x^2, y^2) + ... + p(x^n, y^n)] \end{aligned} \tag{21}$$

where $k \in [0,1)$ with Then F has a unique n-tuple fixed point.

Example 2.1. Let $X = [0,\infty)$ endowed with the usual partial metric p defined $p : X^n \to [0,\infty)$ with $p(x,y) = \max\{x,y\}$. The partial metric space (X,p) is complete because $(X,\tilde{d})$ is the Euclidean metric space which is complete. Consider the mapping $F : X^n \to X$ defined by $F(x^1, x^2, ..., x^n) = \frac{(x^1+x^2+...+x^n)}{4}$. For any $x^1, x^2, ..., x^n \in X$, we have

$$\begin{aligned} & p(F(x^1, x^2, ..., x^n), F(y^1, y^2, ..., y^n)) \\ = \; & \tfrac{1}{4} \; \max\{x^1 + x^2 + ... + x^n, y^1 + y^2 + ... + y^n\} \\ \leq \; & \tfrac{1}{4} \; [\max\{x^1, y^1\} + ... + \max\{x^n, y^n\}] \\ = \; & \tfrac{1}{4} \; [p(x^1, y^1) + ... + p(x^n, y^n)], \end{aligned} \tag{22}$$

which is the contractive condition (21) for $k = \frac{1}{2}$. Therefore, by Corollary 2.1, F has a unique n-tuple fixed point, which is (0,0,...,0).

Theorem 2.2. Let (X,p) be a complete partial metric space. Suppose $F : X^n \to X$ satisfies the following contractive condition for all $x^1, x^2, ..., x^n, y^1, y^2, ..., y^n \in X$

$$\begin{aligned} & p(F(x^1, ..., x^n), F(y^1, ..., y^n)) \\ \leq \; & kp(F(x^1, ..., x^n), x^1) + lp(F(y^1, ..., y^n), y^1) \end{aligned} \tag{23}$$

where $k, l \in [0,1)$ with $k + l < 1$. Then F has a unique n-tuple fixed point.

Proof. We take the same sequences $\{x_m^1\}, \{x_m^2\}, ...$ and $\{x_m^n\}$ given in the proof of Theorem 2.1 by

$$\begin{aligned} & x_{m+1}^1 = F(x_m^1, ..., x_m^n),\; x_{m+1}^2 \\ = \; & F(x_m^2, ..., x_m^n), ..., x_{m+1}^n = F(x_m^n, ..., x_m^1) \end{aligned} \tag{24}$$

Applying (23), we get

$$\begin{aligned} & p(x_m^1, x_{m+1}^1) \leq \delta p(x_m^1, x_{m-1}^1) \\ & p(x_m^2, x_{m+1}^2) \leq \delta p(x_m^2, x_{m-1}^2) \\ & \cdot \\ & \cdot \\ & \cdot \\ & p(x_m^n, x_{m+1}^n) \leq \delta p(x_m^n, x_{m-1}^n) \end{aligned} \tag{25}$$

where $\delta = \frac{k}{1-l}$. By the definition of $\widetilde{d}$, we have

$$\begin{aligned}
\widetilde{d}(x_m^1, x_{m+1}^1) &\leq 2p(x_m^1, x_{m+1}^1) \leq \delta^m p(x_1^1, x_0^1)\\
\widetilde{d}(x_m^2, x_{m+1}^2) &\leq 2p(x_m^2, x_{m+1}^2) \leq \delta^m p(x_1^2, x_0^2)\\
&.\\
&.\\
&.\\
\widetilde{d}(x_m^n, x_{m+1}^n) &\leq 2p(x_m^n, x_{m+1}^n) \leq \delta^m p(x_1^n, x_0^n)
\end{aligned} \tag{26}$$

Since $k + l < 1$, hence $\delta < 1$, so the sequences $\{x_m^1\}, \{x_m^2\}, ...$ and $\{x_m^n\}$ are Cauchy sequences in the metric space $(X, \widetilde{d})$. The partial metric space (X, p) is complete, hence thanks to Lemma 1.1, the metric space $(X, \widetilde{d})$ is complete, so there exist $u^1, u^2, ..., u^n \in X$ such that

$$\lim_{m\to\infty} \widetilde{d}(x_m^1, u^1) = \lim_{m\to\infty} \widetilde{d}(x_m^2, u^2) = ... = \lim_{m\to\infty} \widetilde{d}(x_m^n, u^n) = 0. \tag{27}$$

Again by Lemma 1.1 we get

$$\begin{aligned}
p(u^1, u^1) &= \lim_{m\to\infty} p(x_m^1, u^1) = \lim_{m\to\infty} p(x_m^1, x_m^1),\\
p(u^2, u^2) &= \lim_{m\to\infty} p(x_m^2, u^2) = \lim_{m\to\infty} p(x_m^2, x_m^2),\\
&.\\
&.\\
&.\\
p(u^n, u^n) &= \lim_{m\to\infty} p(x_m^n, u^n) = \lim_{m\to\infty} p(x_m^n, x_m^n).
\end{aligned} \tag{28}$$

But, from condition $(p2)$ and (25),

$$p(x_m^1, x_m^1) \leq p(x_m^1, x_{m+1}^1) \leq \delta^m p(x_1, x_0), \tag{29}$$

so $\lim_{m\to+\infty} p(x_m^1, x_m^1) = 0$. It follows that

$$p(u^1, u^1) = \lim_{m\to\infty} p(x_m^1, u^1) = \lim_{m\to\infty} p(x_m^1, x_m^1) = 0. \tag{30}$$

Similarly, we get

$$\begin{aligned}
p(u^2, u^2) &= \lim_{m\to\infty} p(x_m^2, u^2) = \lim_{m\to\infty} p(x_m^2, x_m^2) = 0,\\
&.\\
&.\\
&.\\
p(u^n, u^n) &= \lim_{m\to\infty} p(x_m^n, u^n) = \lim_{m\to\infty} p(x_m^n, x_m^n) = 0.
\end{aligned} \tag{31}$$

Therefore, we have, using (23),

$$\begin{aligned} p(F(u^1,u^2,...,u^n),u^1) &\leq p(F(u^1,u^2,...,u^n),x^1_{m+1}) + p(x^1_{m+1},u^1) \\ &\leq p(F(u^1,u^2,...,u^n),F(x^1_m,x^2_m,...,x^n_m)) \\ &+ p(x^1_{m+1},u^1) \\ &\leq kp(F(u^1,u^2,...,u^n),u^1) \\ &+ lp(F(x^1_m,x^2_m,...,x^n_m),x^1_m) + p(x^1_{m+1},u^1) \\ &\leq kp(F(u^1,u^2,...,u^n),u^1) + lp(x^1_{m+1},x^1_m) \\ &+ p(x^1_{m+1},u^1) \end{aligned} \tag{32}$$

and letting $m \to +\infty$, then from (29)-(31), we obtain

$$p(F(u^1,u^2,...,u^n),u^1) \leq kp(F(u^1,u^2,...,u^n),u^1).$$

This is a contradiction, so $p(F(u^1,u^2,...,u^n),u^1) = 0$ that is $F(u^1,u^2,...,u^n) = u^1$. Similarly, we have $F(u^2,u^1,...,u^n) = u^2,...$ and $F(u^n,u^{n-1},...,u^1) = u^n$, meaning that $(u^1,u^2,...,u^n)$ is n-tuple fixed point of F. Now, let $(v^1,...,v^n)$ is another tuples fixed point of F, then by using the condition (p2) and (23)

$$\begin{aligned} p(u^1,v^1) &= p(F(u^1,u^2,...,u^n),F(v^1,v^2,...,v^n)) \\ &\leq kp(F(u^1,u^2,...,u^n),u^1) + lp(F(v^1,v^2,...,v^n),v^1) \\ &\leq kp(u^1,u^1) + lp(v^1,v^1) \\ &\leq kp(u^1,v^1) + lp(u^1,v^1) = (k+l)p(u^1,v^1), \end{aligned} \tag{33}$$

that is, $p(u^1,v^1) = 0$ since $k+l < 1$. It follows that $u^1 = v^1$. Similarly, we can have $u^2 = v^2,...$ and $u^n = v^n$.

Theorem 2.3. Let (X,p) be a complete partial metric space. Suppose $F : X^n \to X$ satisfies the following contractive condition for all $x^1,x^2,...,x^n,y^1,y^2,...,y^n \in X$

$$\begin{aligned} p(F(x^1,...,x^n),F(y^1,...,y^n)) \leq\ & kp(F(x^1,...,x^n),y^1) \\ & +lp(F(y^1,...,y^n),x^1) \end{aligned} \tag{34}$$

where $k,l \in [0,1)$ with $k+2l < 1$. Then F has a unique n-tuples fixed point.

Proof. Since $k+2l < 1$, hence $k+l < 1$, and the proof of the uniqueness of n-tuple fixed point in this theorem is trivial. To prove the existence of fixed point, choose the sequences $\{x^1_m\},\{x^2_m\},...$ and $\{x^n_m\}$ like in the proof of Theorem 2.1 as follows:

$$\begin{aligned} x^1_{m+1} = F(x^1_m,...,x^n_m),\ x^2_{m+1} \\ = F(x^2_m,...,x^n_m),...x^n_{m+1} = F(x^n_m,...,x^1_m) \end{aligned} \tag{35}$$

Applying (31), we get

$$\begin{aligned} p(x_m^1, x_{m+1}^1) &= p(F(x_{m-1}^1, ..., x_{m-1}^n), F(x_m^1, ..., x_m^n)) \\ &\leq kp(F(x_{m-1}^1, ..., x_{m-1}^n), x_m^1) + lp(x_{m-1}^1, \\ &\quad F(x_m^1, ..., x_m^n)) \\ &\leq kp(x_{m+1}^1, x_m^1) + lp(x_{m+1}^1, x_{m-1}^1) \\ &\leq kp(x_{m+1}^1, x_m^1) + lp(x_{m+1}^1, x_m^1) \\ &+ lp(x_m^1, x_{m-1}^1) - lp(x_m^1, x_m^1) \\ &\leq (k+l)p(x_{m+1}^1, x_m^1) + lp(x_m^1, x_{m-1}^1) \end{aligned} \tag{36}$$

It follows that for any $n \in N$

$$p(x_m^1, x_{m+1}^1) \leq \frac{k}{1-l-k} p(x_m^1, x_{m-1}^1) \tag{37}$$

Let us take $\eta = \frac{k}{1-l-k}$. Hence, we deduce that

$$\widetilde{d}(x_m^1, x_{m+1}^1) \leq 2p(x_m^1, x_{m+1}^1) \leq 2\eta^m p(x_1^1, x_0^1) \tag{38}$$

By using the condition $0 \leq k + l < 1$, we get that $0 \leq \eta < 1$, so the sequence $\{x_m^1\}$ is Cauchy sequence in the complete metric space $(X, \widetilde{d})$. Of course, similar arguments apply to case of the sequences $\{x_m^2\}, ...$ and $\{x_m^n\}$ in order to prove that

$$\begin{aligned} &\widetilde{d}(x_m^2, x_{m+1}^2) \leq 2p(x_m^2, x_{m+1}^2) \leq 2\eta^m p(x_1^2, x_0^2) \\ &. \\ &. \\ &. \\ &\widetilde{d}(x_m^n, x_{m+1}^n) \leq 2p(x_m^n, x_{m+1}^n) \leq 2\eta^m p(x_1^n, x_0^n) \end{aligned} \tag{39}$$

and, thus, the sequences $\{x_m^2\}, ...$ and $\{x_m^n\}$ are Cauchy sequences in the complete metric space $(X, \widetilde{d})$. Therefore, there exist $u^1, u^2, ..., u^n \in X$ such that

$$\lim_{m\to\infty} \widetilde{d}(x_m^1, u^1)) = \lim_{m\to\infty} \widetilde{d}(x_m^2, u^2)) = ... = \lim_{m\to\infty} \widetilde{d}(x_m^n, u^n) = 0. \tag{40}$$

From Lemma 1.1, we get

$$\begin{aligned} p(u^1, u^1) &= \lim_{m\to\infty} p(x_m^1, u^1) = \lim_{m\to\infty} p(x_m^1, x_m^1), \\ p(u^2, u^2) &= \lim_{m\to\infty} p(x_m^2, u^2) = \lim_{m\to\infty} p(x_m^2, x_m^2), \\ &. \\ &. \\ &. \\ p(u^n, u^n) &= \lim_{m\to\infty} p(x_m^n, u^n) = \lim_{m\to\infty} p(x_m^n, x_m^n). \end{aligned} \tag{41}$$

By $(p2)$ and (39) we obtain that

$$p(x_m^1, x_m^1) \leq p(x_m^1, x_{m+1}^1) \leq \eta^m p(x_1, x_0), \tag{42}$$

taking $n \to \infty$, we obtain $\lim_{m\to+\infty} p(x_m^1, x_m^1) = 0$. It follows that

$$p(u^1, u^1) = \lim_{m\to\infty} p(x_m^1, u^1) = \lim_{m\to\infty} p(x_m^1, x_m^1) = 0. \tag{43}$$

Similarly, we get

$$\begin{aligned} p(u^2, u^2) &= \lim_{m\to\infty} p(x_m^2, u^2) = \lim_{m\to\infty} p(x_m^2, x_m^2) = 0, \\ &. \\ &. \\ &. \\ p(u^n, u^n) &= \lim_{m\to\infty} p(x_m^n, u^n) = \lim_{m\to\infty} p(x_m^n, x_m^n) = 0. \end{aligned} \tag{44}$$

Therefore, we have, using (34)and (p4),

$$\begin{aligned} p(F(u^1, u^2, ..., u^n), u^1) &\le p(F(u^1, u^2, ..., u^n), x_{m+1}^1) + p(x_{m+1}^1, u^1) \\ &\le p(F(u^1, u^2, ..., u^n), F(x_m^1, x_m^2, ..., x_m^n)) \\ &+ p(x_{m+1}^1, u^1) \\ &\le kp(F(u^1, u^2, ..., u^n), x_m^1) \\ &+ lp(F(x_m^1, x_m^2, ..., x_m^n), u^1) + p(x_{m+1}^1, u^1) \\ &\le kp(F(u^1, u^2, ..., u^n), x_m^1) + kp(x_m^1, u^1) \\ &+ lp(x_{m+1}^1, u^1) + p(u^1, x_m^1) + p(x_{m+1}^1, u^1). \end{aligned} \tag{45}$$

Letting $m \to +\infty$, yields, using (44),

$$p(F(u^1, u^2, ..., u^n), u^1) \le kp(F(u^1, u^2, ..., u^n), u^1).$$

This is a contradiction, because $k < 1$, we have $p(F(u^1, u^2, ..., u^n), u^1) = 0$ that is $F(u^1, u^2, ..., u^n) = u^1$. Similarly, from (44) we have $F(u^2, u^1, ..., u^n) = u^2, ...$ and $F(u^n, u^{n-1}, ..., u^1) = u^n$, meaning that $(u^1, u^2, ..., u^n)$ is n-tuples fixed point of F.
When $k = l = \frac{a}{2}$ in Theorem 2.2 and Theorem 2.3, we obtain the following corollaries.

Corollary 2.2. Let (X, p) be a complete partial metric space. Suppose $F : X^n \to X$ satisfies the following contractive condition for all $x^1, x^2, ..., x^n, y^1, y^2, ..., y^n \in X$

$$\begin{aligned} p(F(x^1, ..., x^n), F(y^1, ..., y^n)) &\le \frac{a}{2}[p(F(x^1, ..., x^n), x^1) \\ &+ p(F(y^1, ..., y^n), y^1)] \end{aligned} \tag{46}$$

where $a \in [0, 1)$. Then F has a unique n-tuples fixed point.

Corollary 2.3. Let (X, p) be a complete partial metric space. Suppose $F : X^n \to X$ satisfies the following contractive condition for all $x^1, x^2, ..., x^n, y^1, y^2, ..., y^n \in X$

$$\begin{aligned} p(F(x^1, ..., x^n), F(y^1, ..., y^n)) &\le \frac{a}{2}[p(F(x^1, ..., x^n), y^1) \\ &+ p(F(y^1, ..., y^n), x^1)] \end{aligned} \tag{47}$$

where $a \in [0, 1)$. Then F has a unique n-tuples fixed point.

3 Comments

In this section we show that n-tuples or generalized coupled fixed point equation if and only if fixed point equation:

Definition 3.1. Suppose that X^n be a nonempty set and let $G : X^n \to X^n$ be a mapping, we say that G has a fixed point $(u^1, u^2, ..., u^n)$ if $G(u^1, u^2, ..., u^n) = (u^1, u^2, ..., u^n)$.

Lemma 3.1. Suppose that (X, p) be a partial metric space and let $\hat{p} : X^n \times X^n \to [0, \infty)$ defined by (1).
Then $(X^n, \hat{p})$ is a partial metric space.

Theorem 3.1. Suppose that $(X^n, \hat{p})$ be a complete partial metric space and let $G : X^n \to X^n$, be a contraction mapping defined by

$$\hat{p}(G(x^1, x^2, ..., x^n), G(y^1, y^2, ..., y^n)) \leq k\, \hat{p}((x^1, x^2, ..., x^n), (y^1, y^2, ..., y^n)). \tag{48}$$

where $0 \leq k < 1$. Then G has a unique fixed point in X^n.

Proof. The proof follow directly by Banach contraction mapping for partial metric spaces.

Lemma 3.2. Suppose that (X, p) be a partial metric space and let $\widetilde{p} : X^2 \times X^2 \to [0, \infty)$ defined by

$$\widetilde{p}((x, y), (u, v)) = kp(x, u) + lp(y, v), \tag{49}$$

where $x, y, u, v \in X$, $k, l \in [0, 1)$ with $k + l < 1$.
Then $(X^2, \widetilde{p})$ is a partial metric space.

Theorem 3.2. Suppose that $(X^2, \widetilde{p})$ be a complete partial metric space and let $\widetilde{G} : X^2 \to X^2$, be a contraction mapping defined by

$$\widetilde{p}(\widetilde{G}(x, y), \widetilde{G}(u, v)) \leq k\, \widetilde{p}((x, y), (u, v)).$$

where $0 \leq k < 1$. Then $\widetilde{G}$ has a unique fixed point in X^2, i.e., $\widetilde{G}(u, v) = (u, v)$.

Lemma 3.3. Suppose that (X, d) be a metric space and let $d^3 : X^3 \times X^3 \to [0, \infty)$ defined by

$$d^3((x, y, z), (u, v, w)) = jd(x, u) + kd(y, v) + ld(z, w),$$

where $x, y, z, u, v, w \in X$ and $j, k, l \in [0, 1)$, $j + k + l < 1$.
Then (X^3, d^3) a metric space.

Theorem 3.3. Suppose that (X^3, d^3) be a complete metric space and let $F^3 : X^3 \to X^3$, be a contraction mapping defined by

$$d^3(F^3(x,y,z), F^3(u,v,w)) \leq k\, d^3((x,y,z),(u,v,w)), \quad 0 \leq k < 1.$$

Then F^3 has a unique fixed point in X^3.

Lemma 3.4. Suppose that (X, d) be a metric space and let $d^2 : X^2 \times X^2 \to [0, \infty)$ be a distance function defined by

$$d^2((x,y),(u,v)) = \frac{k}{2}[d(x,u) + d(y,v)], \quad 0 \leq k < 1,$$

where $(x, y), (u, v) \in X \times X$.
Then (X^2, d^2) a metric space.

Theorem 3.4. Suppose that (X^2, d^2) be a complete metric space and let $F^2 : X^2 \to X^2$, be a contraction mapping defined by

$$d^2(F^2(x,y), F^2(u,v)) \leq k\, d^2((x,u),(y,v)).$$

Then F^2 has a unique fixed point in X^2, i.e., $F^2(u, v) = (u, v)$.

Remark 3.1. (1) Theorem 3.1 $\Leftrightarrow$ Theorem 2.1,
(2) Theorem 3.2 $\Leftrightarrow$ Theorem 1.3 [Theorem 2.1, [2]],
(3) Theorem 3.3 $\Leftrightarrow$ Theorem 1.2 [Theorem 7, [3]],
(4) Theorem 3.4 $\Leftrightarrow$ Theorem 1.1 [Theorem 2.1, [4]].
We mean by "$\Leftrightarrow$" that is "if and only if".

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ADDITIVE MAPPINGS OF SEMIPRIME RINGS WITH INVOLUTION

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Abstract

Let R be a semiprime ring with involution '$*$' and let $T : R \to R$ be an additive mapping satisfying any one of the following conditions: (i) $2T(x^{n+1}) = T(x)(x^*)^n + (x^*)^n T(x)$, (ii) $T(xyx) = x^*T(y)x^*$ and (iii) $3T(xyx) = T(x)y^*x^* + x^*T(y)x^* + x^*y^*T(x)$ for all $x, y \in R$. Then $T(xy) = T(y)x^*$ for all $x, y \in R$.

1 Introduction

Throughout R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, where $n > 1$ is an integer, in case $nx = 0$, $x \in R$, implies $x = 0$. As usual, the commutator $xy - yx$ will be denoted by $[x, y]$. Recall that a ring R is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies $a = 0$. An additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. Every derivation on R is a Jordan derivation but the converse need not be true in general. A classical result due to Herstein [8, Theorem 3.3] , asserts that a Jordan derivation on a prime ring of characteristic different from two is a derivation. A brief proof of this result can be found in [5]. Further, Cusack [5] extended Herstein's theorem for 2-torsion free semiprime ring (see also [4] for an alternate proof). An additive mapping $T : R \to R$ is called a left (right) centralizer in case $T(xy) = T(x)y$ $(T(xy) = xT(y))$ holds for all $x, y \in R$. Following Zalar [13] T is called a centralizer if T is both a left and a right centralizer. If R has an identity element, $T : R \to R$ is left (right) centralizer iff

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T is of the form $T(x) = ax$ ($T(x) = xa$) for some fixed $a \in R$. An additive mapping $T : R \to R$ is called a left Jordan centralizer (resp. right Jordan centralizer) in case $T(x^2) = T(x)x$ (resp. $T(x^2) = xT(x)$) holds for all $x \in R$. Following ideas form Bresar [4], Zalar [13] proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. An additive mapping $D : R \to R$, where R is an arbitrary ring, is a Jordan triple derivation, if $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ holds for all pairs $x, y \in R$. One can easily prove that any Jordan derivation on a 2-torsion free ring is a Jordan triple derivation (see [4]), but not conversely. The converse of the above problem was explored by Bresar [4] who proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Inspired by this result Vukman [9] proved the following result:

Theorem 1.1 ([9, Theorem 1]). *Let R be a 2-torsion free semiprime ring and $T : R \to R$ be an additive mapping satisfying $T(xyx) = xT(y)x$ for all $x, y \in R$. Then in this case T is a centralizer.*

Obviously, any centralizer $T : R \to R$, where R is an arbitrary ring, satisfies the relation $T(xyx) = xT(y)x$ for all $x, y \in R$, which means that Theorem 1.1 characterizes centralizers among all additive mappings in 2-torsion free semiprime rings.

An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(x^*)^* = x$ and $(xy)^* = y^*x^*$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*$-ring. Let R be a ring with involution '$*$'. An additive mapping $T : R \to R$ is called left (right) $*$-centralizer, if $T(xy) = T(x)y^*$ ($T(xy) = x^*T(y)$) holds for all $x, y \in R$ and $T : R \to R$ is called left (right) Jordan $*$-centralizer, if $T(x^2) = T(x)x^*$ ($T(x^2) = x^*T(x)$) for all $x \in R$. If T is both left as well as right Jordan $*$-centralizer of R, then it is called Jordan $*$-centralizer of R. For any fixed element $a \in R$ the mapping $T(x) = ax^*$ ($T(x) = x^*a$) is left (right) Jordan $*$-centralizer.

Let $T : R \to R$ be an additive mapping satisfying

$$T(xyx) = x^*T(y)x^* \text{ for all } x, y \in R. \tag{1.1}$$

In view of Theorem 1.1 it is natural to ask whether the additive mapping satisfying (1.1) is left (right) Jordan $*$-centralizer. The present paper deals with the study of similar kinds of problems involving additive mappings in semiprime rings. In fact, it is shown that if an additive mapping T on a 2-torsion free semiprime ring R satisfies (1.1), then $T(xy) = T(y)x^*$ ($T(xy) = y^*T(x)$) for all $x, y \in R$. Further, it is also shown that similar conclusion holds when the underlying ring R satisfies the property $3T(xyx) = T(x)y^*x^* + x^*T(y)x^* + x^*y^*T(x)$ for all $x, y \in R$.

2 Main Results

We begin our discussion with the following theorem which is motivated by Theorem 2 of Vukman and Kosi-Ulbl [11].

Theorem 2.1. *Let $n \geqslant 2$ be a fixed integer and let R be a $2n$-torsion free semiprime $*$-ring with identity element. Suppose that there exists an additive mapping $T : R \to R$ such that $2T(x^{n+1}) = T(x)(x^*)^n + (x^*)^n T(x)$ for all $x \in R$. Then $T(xy) = T(y)x^*$ for all $x, y \in R$.*

For developing the proof of the above theorem we need the following result:

Proposition 2.1. *Let R be a 2-torsion free semiprime ring with involution $*$. Suppose that $T : R \to R$ is an additive mapping satisfying $T(x^2) = T(x)x^*$ for all $x \in R$. Then $T(xy) = T(y)x^*$ for all $x, y \in R$.*

Proof. Given that

$$T(x^2) = T(x)x^* \text{ for all } x \in R. \tag{2.1}$$

Let us introduce a function S on R by the relation $S(x) = T(x^*)$ for all $x \in R$. Replacing x by x^2, we get $S(x^2) = T({x^*}^2)$ for all $x \in R$. Therefore, we have an additive mapping $S : R \to R$ satisfying the relation

$$S(x^2) = T({x^*}^2) = T(x^*)x = S(x)x \text{ for all } x \in R.$$

Hence, S is a left Jordan centralizer. It follows from the result of Zalar [13] that S is a left centralizer. Now we have $T(xy) = S(y^*x^*) = S(y^*)x^* = T(y)x^*$ for all $x, y \in R$, which completes the proof.

Proof of Theorem 2.1. We have

$$2T(x^{n+1}) = T(x)(x^*)^n + (x^*)^n T(x) \text{ for all } x \in R. \tag{2.2}$$

Similarly, as in the proof of Proposition 2.1, we introduce a function S on R by $S(x) = T(x^*)$ for all $x \in R$. Now, $2S(x^{n+1}) = 2T((x^*)^{n+1})$ for all $x \in R$. Then, by (2.2), we get for all $x \in R$

$$\begin{aligned} 2S(x^{n+1}) &= T(x^*)x^n + x^n T(x^*) \\ &= S(x)x^n + x^n S(x). \end{aligned}$$

Then, by Theorem 2 of [11], S is a Jordan left centralizer. Therefore, $S(x^2) = S(x)x$ for all $x \in R$. Now, using main theorem of Zalar [13], S is a left centralizer i.e., $S(xy) = S(x)y$ for all $x, y \in R$. Hence, using the same techniques as used in the proof of Proposition 2.1, we get the required result.

Theorem 2.2. *Let R be a 2-torsion free semiprime ring with involution $*$. If $T : R \to R$ is an additive mapping satisfying $T(xyx) = x^*T(y)x^*$ for all $x, y \in R$, then $T(xy) = T(y)x^* = y^*T(x)$ holds for all $x, y \in R$.*

Proof. By hypothesis

$$T(xyx) = x^*T(y)x^* \text{ for all } x, y \in R. \tag{2.3}$$

As in the case of Proposition 2.1, we introduce a mapping S on R such that $S(x) = T(x^*)$ for all $x \in R$. We have $S(xyx) = T\big((xyx)^*\big) = T(x^*y^*x^*)$ for all $x, y \in R$. Then, using our hypothesis, we find that for all $x, y \in R$

$$\begin{aligned} S(xyx) &= xT(y^*)x \\ &= xS(y)x. \end{aligned}$$

Hence, S satisfies all the requirements of Theorem 1.1 and therefore S is a two-sided centralizer. Using the same techniques as we have used in previous theorem, we get the required result.

Corollary 2.1. *Let R be a prime ring with char$R \neq 2$ and involution $*$ and let $T : R \to R$ be a nonzero left Jordan $*$-centralizer. If $T(x) \in Z(R)$ holds for all $x \in R$, then R is commutative.*

Proof. By hypothesis, we have that $[T(x), y] = 0$ for all $x, y \in R$. Replacing x by x^2, we have $[T(x^2), y] = 0$ for all $x, y \in R$ i.e., $[T(x)x^*, y] = 0$ for all $x, y \in R$. This implies that $T(x)[x^*, y] + [T(x), y]x^* = 0$ for all $x, y \in R$. Since by hypothesis $[T(x), y] = 0$ for all $x, y \in R$, we get $T(x)[x^*, y] = 0$ for all $x, y \in R$. Replace y by yz, to get $T(x)y[x^*, z] = 0$ for all $x, y, z \in R$. Therefore, $T(x)R[x^*, z] = \{0\}$ for all $x, z \in R$.

Thus, the primeness of R and the fact that $(R, +)$ is not the union of two of its proper subgroups show that either $T(x) = 0$ for all $x \in R$ or $[x^*, z] = 0$ for all $x, z \in R$. But since $T \neq 0$, we find that $[x^*, z] = 0$ or $[x, z] = 0$ for all $x, z \in R$ i.e., R is commutative.

The main theorem of Vukman and Kosi-Ulbl [11, Theorem 1] was extended by Ashraf et. al. [3, Theorem 2.3] as follows: an additive mapping T on a 2-torsion free semiprime $*$-ring satisfying $2T(xyx) = T(x)\alpha(y^*x^*) + \alpha(x^*y^*)T(x)$ for all $x, y \in R$ and automorphism α, is a Jordan α-*centralizer of R i.e., R satisfies $T(x^2) = \alpha(x^*)T(x)$ and $T(x^2) = T(x)\alpha(x^*)$ for all $x \in R$. In view of this result for $\alpha = I$ and Proposition 2.1, we obtain the following result:

Theorem 2.3. *Let R be a 2-torsion free semiprime $*$-ring. Suppose $T : R \to R$ is an additive mapping satisfying $2T(xyx) = T(x)y^*x^* + x^*y^*T(x)$ for all $x, y \in R$. Then $T(xy) = T(y)x^*$ for $x, y \in R$.*

Further, motivated by the work of Bresar [4], Vukman and Kosi-Ulbl obtained the following result:

Theorem 2.4 ([12, Theorem 1]). *Let R be a 2-torsion free semiprime ring and $T : R \to R$ be an additive mapping satisfying $3T(xyx) = T(x)yx + xT(y)x + xyT(x)$ for all $x, y \in R$. Then there exists an element $\lambda \in C$, the extended centroid of R such that $T(x) = \lambda x$ for all $x \in R$.*

Inspired by the above theorem, we prove the following result for semiprime $*$-ring:

Theorem 2.5. *Let R be a 2-torsion free semiprime ring with involution $*$ and $T : R \to R$ be an additive mapping satisfying $3T(xyx) = T(x)y^*x^* + x^*T(y)x^* + x^*y^*T(x)$ for all $x, y \in R$. Then $T(xy) = T(y)x^*$ for all $x, y \in R$.*

Proof. Given that

$$3T(xyx) = T(x)y^*x^* + x^*T(y)x^* + x^*y^*T(x) \text{ for all } x, y \in R. \tag{2.4}$$

Let us introduce a mapping S on R such that $S(x) = T(x^*)$ for all $x \in R$. Then, for all $x, y \in R$

$$\begin{aligned} 3S(xyx) &= 3T(x^*y^*x^*) \\ &= T(x^*)yx + xT(y^*)x + xyT(x^*) \\ &= S(x)yx + xS(y)x + xyS(x). \end{aligned}$$

Hence, by Theorem 2.4, there exists $\lambda \in C$ such that $S(x) = \lambda x$ for all $x \in R$. Therefore,

$$\begin{aligned} T(xy) &= S(y^*x^*) \\ &= \lambda y^*x^* \\ &= (\lambda y^*)x^* \\ &= S(y^*)x^* \\ &= T(y)x^* \text{ for all } x, y \in R. \end{aligned}$$

This gives the required result.

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Citation in the text should be numbered in square brackets, for example [1], or, Lopez-Bonilla and Ovando [1], referring to an alphabetically ordered and numbered list. If a work of more than three authors or, editors are cited, only the first author's name followed by 'et al.' need be given, for example, Ahmad et al. [1].

In the list of reference, the following examples should be observed:

[1] Cenzig, B. : *A generalization of the Banach-Stone theorem,* Proc. Amer. Math. Soc. 40 (1973) 426-430.

[2] Blair, D. : *Contact Manifolds in Riemannian Geometry,* Lecture Notes in Mathematics, Vol. 509, Springer-Verlag, Berlin, 1976.

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